FOCK SPACE & CREATION/ANNIHILATION OPERATORS

Fock space is the Hilbert space for describing systems with varying particle numbers (e.g. quantum field theories).
It is also convenient for studying many-particle systems with fixed number of particles.

DEF: <u>Fock space</u> over a Hilbert space h is defined as

$$\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} h^{\otimes n} .$$

By this direct sum we mean the vector space of sequences

$$\psi = (\psi_n)_{n \geq 0},$$

where $\psi_0 \in \mathbb{C}$ and $\psi_n \in h^{\otimes n}$ $(n \geq 1)$,
such that the norm

$$\|\psi\|_{\mathcal{F}} := \Big(\sum_{n=0}^{\infty} \|\psi_n\|^2_{h^{\otimes n}} \Big)^{1/2} \text{ is finite.}$$

$\mathcal{F}$ is a Hilbert space with the scalar product

$$\langle \psi, \phi \rangle_{\mathcal{F}} := \sum_{n \geq 0}^{\infty} \langle \psi_n, \varphi_n \rangle_{h^{\otimes n}} .$$

DEF: <u>Number operator:</u> $\mathcal{N}\psi = (n\psi_n)_{n \geq 0}$,

$$D(\mathcal{N}) = \{ \psi \in \mathcal{F} : (n\psi_n)_{n\geq 0} \in \mathcal{F} \} = \{ \psi \in \mathcal{F} : \sum_{n \geq 1} n^2 \|\psi_n\|^2_{h^{\otimes n}} < \infty \}.$$

LEMMA: $\mathcal{N}$ is selfadjoint.

DEF: Let $S_n = \{\text{permutations of } \{1, \dots, n\}\}$.
For $\sigma \in S_n$, let P_σ be the unitary

compare to lecture on bosons and fermions

$$P_\sigma(\varphi_1 \otimes \dots \otimes \varphi_n) := \varphi_{\sigma(1)} \otimes \dots \otimes \varphi_{\sigma(n)}.$$

Define <u>symmetrization</u> and <u>antisymmetrization</u> operators:

$$S_n := \frac{1}{n!} \sum_{\sigma \in S_n} P_\sigma \,, \qquad A_n := \frac{1}{n!} \sum_{\sigma \in S_n} \underbrace{\mathrm{sgn}(\sigma)}_{\text{sign of the permutation.}} P_\sigma .$$

Now on all of Fock space:

$$\begin{aligned} S : \mathcal{F} &\longrightarrow \mathcal{F} \\ (\psi_n)_{n \geq 0} &\longmapsto (S_n \psi_n)_{n \geq 0} \end{aligned}$$

and

$$\begin{aligned} A : \mathcal{F} &\longrightarrow \mathcal{F} \\ (\psi_n)_{n \geq 0} &\longmapsto (A_n \psi_n)_{n \geq 0} . \end{aligned}$$

$\mathrm{ran}(A) = A\mathcal{F} =: \mathcal{F}_a$ is called <u>fermionic Fock space</u>,
$\mathrm{ran}(S) = S\mathcal{F} =: \mathcal{F}_s$ is called <u>bosonic Fock space</u>.

DEF: $\Omega := (1, 0, 0, 0, \dots) \in \mathcal{F}_s \cap \mathcal{F}_a \subset \mathcal{F}$ is called <u>vacuum vector</u>.
It has zero particles: $\mathcal{N}\Omega = 0$.

DEF: Let $f \in h$. We define
$$b(f) : h^{\otimes n} \longrightarrow h^{\otimes n-1}$$
$$b^*(f) : h^{\otimes n} \longrightarrow h^{\otimes n+1}$$
as acting by:
$$b(f)\Omega = 0 \,, \quad b^*(f)\Omega = f \,,$$
$$b(f)(e_1 \otimes \dots \otimes e_n) = \langle f, e_n \rangle \, e_1 \otimes \dots \otimes e_{n-1}$$
$$b^*(f)(e_1 \otimes \dots \otimes e_n) = e_1 \otimes \dots \otimes e_n \otimes f .$$
By linear extension this defines $b(f)$ and $b^*(f)$ on Fock space $\mathcal{F}$.

LEMMA: $\|b(f)\| \leq \|f\|$, $\|b^*(f)\| \leq \|f\|$.

As bounded operators, $b(f)$ and $b^*(f)$ can be extended to all of Fock space.

REMARK: $f \mapsto b^*(f)$ is linear, but $f \mapsto b(f)$ antilinear.

DEF: Bosonic annihilation operator: $a(f) := b(f) \mathcal{N}^{1/2}$
" creation " : $a^*(f) := \mathcal{N}^{1/2} S\, b^*(f)$.

Same for fermions with A instead of S.

LEMMA: (i) $\mathcal{F}_s$ and $\mathcal{F}_a$ are invariant under the respective creation and annihilation operators.

(ii) $\langle a(f)\varphi, \psi \rangle = \langle \varphi, a^*(f)\psi \rangle$
$\forall \varphi, \psi \in \mathcal{F}_0 = \{\psi \in \mathcal{F}:$ only finitely many ψ_n are $\neq 0\}$.

(iii) <u>Canonical commutation relations:</u> on $\mathcal{F}_0$:

<u>Bosonic operators</u>:

$$[a(f), a^*(g)] := a(f)a^*(g) - a^*(g)a(f)$$
"Commutator"
$$= \langle f, g \rangle_{\mathfrak{h}} \mathbb{1}_{\mathcal{F}}$$
$$[a(f), a(g)] = 0 = [a^*(f), a^*(g)]$$

<u>Canonical anticommutation relations:</u>

<u>Fermionic operators</u>:

$$\{a(f), a^*(g)\} := a(f)a^*(g) + a^*(g)a(f)$$
"anticommutator"
$$= \langle f, g \rangle_{\mathfrak{h}} \mathbb{1}_{\mathcal{F}}$$
$$\{a(f), a(g)\} = 0 = \{a^*(f), a^*(g)\}.$$

(v) bosonic operators are unbounded, but at least $\|a(f)\psi\| \le \|f\| \, \|\mathcal{N}^{1/2}\psi\|$
$$\|a^*(f)\psi\| \le \|f\| \|(\mathcal{N}+1)^{1/2}\psi\|.$$
fermionic operators are bounded:
$$\|a(f)\| \le \|f\|, \quad \|a^*(f)\| \le \|f\|.$$

PROOF: (i) clear by definition.

(ii) Let $\varphi \in S(\mathfrak{h}^{\otimes n+1}) \subset \mathcal{F}_S$, $\psi \in S(\mathfrak{h}^{\otimes n})$.
Then
$$\langle a(f)\varphi, \psi\rangle = \langle b(f)\mathcal{N}^{1/2}\varphi, \psi\rangle$$
$$= \sqrt{n+1}\int dx_1 \dots dx_n \, \overline{\int dx_{n+1} \, \overline{f}(x_{n+1})\varphi(x_1,\dots x_n, x_{n+1})} \times \psi(x_1,\dots x_n)$$
$$= \sqrt{n+1}\int dx_1 \dots dx_{n+1} \, \overline{\varphi(x_1,\dots x_{n+1})}\, \psi(x_1,\dots x_n) f(x_{n+1}).$$

Compare to:
$$\langle \varphi, a^*(f)\psi\rangle = \langle \varphi, \mathcal{N}^{1/2} S \, b^*(f)\psi\rangle$$

using $S = S^*$ (CHECK!)
$$= \sqrt{n+1}\, \langle S\varphi, b^*(f)\psi\rangle$$

$S^2 = S$ (CHECK!) and $\varphi \in \operatorname{ran} S$
$$= \sqrt{n+1}\, \langle \varphi, b^*(f)\psi\rangle$$
$$= \sqrt{n+1}\, \langle \varphi, \psi \otimes f\rangle$$
$$= \sqrt{n+1}\int dx_1 \dots dx_{n+1} \overline{\varphi(x_1 \dots x_{n+1})}\, \psi(x_1,\dots x_n) f(x_{n+1}).$$

(iii) CAR, CCR: important exercise! Check at least $[a(f), a^*(g)]$!

(iv)
$$\|a^*(f)\psi\|^2 = \langle \psi, a(f)a^*(f)\psi\rangle$$
$$\overset{\text{CAR}}{=} \langle \psi, [\langle f, f\rangle - a^*(f)a(f)]\psi\rangle$$
$$= \|f\|_{\mathfrak{h}}^2 \|\psi\|^2 - \underbrace{\|a(f)\psi\|^2}_{\ge 0} \le \|f\|^2 \|\psi\|^2.$$

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