THM 4.3: (Nelson) Let $\mathcal{U}: \mathbb{R} \to \mathcal{L}(\mathfrak{H})$ a SCUG with generator A. If $X \subset D(A)$ is a dense subspace of $\mathfrak{H}$, and invariant under $U(t)$ (i.e. $U(t)X \subset X \ \forall t \in \mathbb{R}$), then $A = \overline{A \restriction_X}$.

$\rightsquigarrow$ So a subspace X is sufficient to calculate generators.

PROOF: Consider the SE : $\begin{cases} i \frac{d}{dt} \varphi(t) = (A\restriction_X)\varphi(t) \\ \varphi(0) = \eta \end{cases}$

It is solved by $U(t)\eta$. Apply Prop. 2.11. ∎

TRANSLATIONGROUP AND MOMENTUM OPERATOR

$\mathfrak{H} := L^2(\mathbb{R})$, $[U(t)\varphi](x) = \varphi(x-t)$. Translation by t.

$X := C_0^\infty(\mathbb{R})$.

Clearly: $\overline{X} = \mathfrak{H}$ and $U(t)X \subset X$.

For $\varphi \in X$: $i \frac{d}{dt}[U(t)\varphi](x)\Big|_{t=0} = i \frac{d}{dt}\varphi(x-t)\Big|_{t=0} = -i\varphi'(x)$.

$\rightsquigarrow$ Conjectured generator $A\restriction_X$: $-i\frac{d}{dx} =: B$.

as required by the def. of the generator!

We still have to show that this is the derivative w.r.t. the L^2-norm!

For $\varphi \in X$: $\left\| \frac{i}{t}[U(t)\varphi - \varphi] - B\varphi \right\|^2 = \int \underbrace{\left| \frac{i}{t}[\varphi(x-t) - \varphi(x)] + i\varphi'(x) \right|^2}_{\text{pointwise} \to 0 \text{ as } t \to \infty} dx$

Take R large enough that $\mathrm{supp}(\varphi) \subset B_{R-1}(0)$. w.l.o.g. $|t| < 1$.

Then: $\left.\begin{array}{ll} \left|\frac{i}{t}[\varphi(x-t) - \varphi(x)]\right| & \leq \|\varphi\|_{L^\infty} \mathbb{1}_{B_R(0)} \\ |\varphi'(x)| & \leq \|\varphi'\|_{L^\infty} \mathbb{1}_{B_R(0)} \end{array}\right\}$ integrable dominating functions.

Now by Lebesgue's dominated convergence theorem:

$\left\| \frac{i}{t}[U(t)\varphi - \varphi] - B\varphi \right\| \longrightarrow 0 \quad (t \to \infty)$.

$\Rightarrow X \subset D(A)$, $A\restriction_X = B$.

By Nelson: $p = \overline{A\restriction_X} = \overline{-i\frac{d}{dx}\restriction_{C_0^\infty(\mathbb{R})}}$ generates U, and $p = p^*$.

Notice also: $-i\frac{d}{dx}\restriction_{C_0^\infty(\mathbb{R})}$ agrees with the def. through the Fourier transform.

$\Rightarrow$ The momentum operator is the generator of translations.

ROTATIONS & ANGULAR MOMENTUM:

$\mathcal{H} := L^2(\mathbb{R}^3)$.

Every rotation $R \in SO(3)$ defines a unitary $U(R)$ by: $U(R)\varphi(x) = \varphi(R^{-1}x)$.

We have $U(R_1)U(R_2) = U(R_1 R_2)$ (this is a group representation).

Choose a rotation axis $e \in \mathbb{R}^3$ and let $t \in \mathbb{R}$ be the rotation angle:

vector product

Check that in $\mathbb{R}^3$: $\frac{d}{dt} R_t x \Big|_{t=0} = e \wedge x$.

Let $U(t) := U(R_t)$, $X := C_0^\infty(\mathbb{R}^3)$.

X is dense in $\mathcal{H}$ and invariant under $U(t)$.

For $\varphi \in X$:

$$
\begin{aligned}
& i\frac{d}{dt}[U(t)\varphi](x)\Big|_{t=0} \\
&= i\frac{d}{dt}\varphi(R_{-t}x)\Big|_{t=0} \\
&= -i\nabla\varphi(x) \cdot (e \wedge x) \\
&= e \cdot (x \wedge -i\nabla\varphi(x)) \\
&= (e \cdot L)\varphi(x) \qquad \text{where } L := x \wedge p = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}.
\end{aligned}
$$

euclidean scalar product

angular momentum operators

Check that this is actually the derivative in the L^2-norm!

L_1, L_2, L_3 are essentially s.a. on $C_0^\infty(\mathbb{R}^3)$.

The operator $e \cdot L$ generates rotations around e-axis.

We have seen that every SCUG has a selfadjoint generator.
Now we show: every selfadjoint operator generates a SCUG.
In particular: If $H = H^*$, then the SE $i\frac{d}{dt}\psi(t) = H\psi(t)$ has a global solution, namely $\psi(t) = U(t)\psi$, where $U(t)$ is generated by H.

THM 4.4: Every selfadjoint A is generator of a unique SCUG.
This SCUG is denoted $U(t) =: e^{-iAt}$.

SKETCH OF PROOF OF 4.4:

1) Approximate A by bounded operators:

Let $B_m := im(A+im)^{-1}$ for $m \in \mathbb{Z}$.

We have
$$B_m\varphi = (im + A - A)(A+im)^{-1}\varphi$$
$$= \varphi - \underbrace{(A+im)^{-1}A\varphi}_{\to 0 \ (|m|\to\infty)} \longrightarrow \varphi,$$
so $\operatorname*{s-lim}_{|m|\to\infty} B_m = \mathbb{1}$.

Let $A_m := B_m A B_{-m}$.

Properties of this regularization:

- $A_m\varphi \underset{(m\to\infty)}{\longrightarrow} A\varphi \quad \forall \varphi \in D(A)$
- since $B_m : \mathcal{H} \to D(A)$,
  A_m is closed and everywhere defined
  $\Rightarrow A_m$ is bounded (by closed graph theorem)
- since $B_m^* = B_{-m}$, we have $A_m^* = A_m$.

Now since A_m is bounded we can just use the ordinary exponential (the power series), and hope to take $m \to \infty$ afterwards.

2) Define $U_m(t) := e^{-iA_m t} := \sum_{k\geq 0} \frac{(-itA_m)^k}{k!}$

(well-defined because A_m is bounded).

$U_m(t)$ is a SCUG, and the s-limit exists: $\underset{m\to\infty}{\text{s-lim}}\, U_m(t) =: U(t)$. Assignment 2

Remains to show: this candidate has the right properties.

Assignment 2: $U(t)$ is a SCUG, and generated by A. ∎

DEF: A (possibly unbounded) operator $B: D \subset \mathcal{H} \to \mathcal{H}$ and a bounded (!) $C: \mathcal{H} \to \mathcal{H}$ <u>commute</u> if $CD \subset D$ and $CB\varphi = BC\varphi \ \forall \varphi \in D$, written: $CB \subset BC$.

RMK: How to define "commute" if both op. are unbounded?
General trick: Use the SCUG or the resolvent, the „natural ways" of making an op. into a bounded operator.
(SCUG or resolvent is usually a matter of choice, we see that both are equiv.)
But it comes at the cost of working only for selfadjoint operators.

DEF: Two selfadjoint operators B, C <u>commute</u> if one of the following conditions is satisfied:

(i) $e^{iCs} e^{iBt} = e^{iBt} e^{iCs} \quad \forall s,t \in \mathbb{R}$

(ii) $e^{iCs} B \subset B e^{iCs} \quad \forall s \in \mathbb{R}$

(iii) $e^{iCs} R_\mu(B) = R_\mu(B) e^{iCs} \quad \forall \mu \in \mathbb{C}\setminus\mathbb{R},\ s \in \mathbb{R}$

(iv) $C R_\mu(B) \supset R_\mu(B)\, C \quad \forall \mu \in \mathbb{C}\setminus\mathbb{R}$

(v) $R_\lambda(C) R_\mu(B) = R_\mu(B) R_\lambda(C) \quad \forall \lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$.

PROP. 4.5: The conditions (i) – (iv) are all equivalent.

PROOF: from e^{iBt} to B by differentiating;
from e^{iCs} to $R_\lambda(C)$ and back using $(C-\lambda)^{-1}\varphi = i\int_0^\infty e^{i\lambda t} e^{-iCt}\varphi\, dt$
(see Thm. 4.1). ∎

RMK: Formal commutators "$[B,C] = BC - CB$" can go very wrong here.
(But often they are a good non-rigorous first attempt.)

DEF: Consider the SE $i\frac{d}{dt}\psi(t) = H\psi(t)$, $H = H^*$, with $\psi(0) = \psi$.

- A selfadjoint operator A is called <u>conserved</u> if $e^{iHt} D(A) \subset D(A)$ and $\langle \psi(t), A\psi(t)\rangle = \langle \psi, A\psi\rangle$ $\forall t \in \mathbb{R}$ $\forall \psi \in D(A)$.
  "it's value does not change in time"
- The operator A <u>generates a symmetry</u> if $e^{iAs} H \subset H e^{iAs}$ $\forall s \in \mathbb{R}$. "the corresponding unitary does not change the Hamiltonian".

THM. 4.6: (Quantum Noether Theorem, conserved quantities generate symmetries)
The following are equivalent:
(i) The operator A is a conserved quantity
(ii) $e^{iHt} A \subset A e^{iHt}$
(iii) The operator A generates a symmetry.

PROOF: trivial by 4.5. ∎

EXAMPLES: 1) Let $V \in L^2 + L^\infty$, real-valued and dep. only on $|x|$, i.e. $\exists f: \mathbb{R} \to \mathbb{R}$ s.th. $V(x) = f(|x|)$.
Then $H = -\Delta + V$ is <u>rotationally symmetric</u>, i.e. for $[U(R)\psi](x) = \psi(R^{-1}x)$: $U(R)H \subset HU(R)$ and the angular momentum operators L_1, L_2, L_3 are conserved.

2) For $H = -\Delta$, the momentum operators $p_j = -i\frac{\partial}{\partial x_j}$, $j = 1,2,3$, are conserved.
(Check one of the conditions! In Fourier space it is easy; the Fourier transform "diagonalizes" both operators, $-\Delta$ and p, simultaneously.)

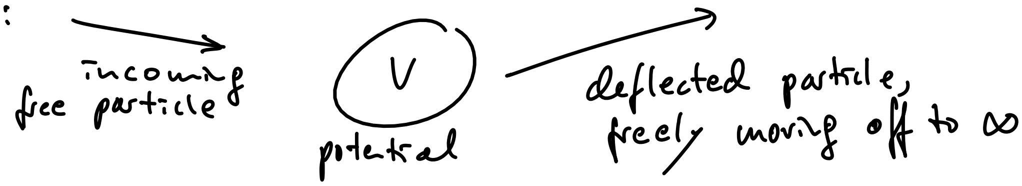
3) For any $H = H^*$, the <u>total energy</u> H is trivially conserved.

V SCATTERING THEORY

[Teschl Ch. 12]

Concerns asymptotic dynamics of particles for $t \to \pm\infty$.
Is a case of perturbation theory: „small" changes to a qualitatively unchanged behaviour.

Typical situation:



Question: What is the relation between the incoming state and the outgoing state?

FREE EVOLUTION: $H_0 = -\Delta/2 = \mathcal{F}^{-1} T_f \mathcal{F}, \quad f(p) = p^2/2.$

$$e^{-iH_0 t} = \mathcal{F}^{-1} e^{-iT_f t} \mathcal{F} = \mathcal{F}^{-1} T_{e^{-ift}} \mathcal{F}$$

[Check this: $i\frac{d}{dt} \mathcal{F}^{-1} T_{e^{-ift}} \mathcal{F} \varphi \big|_{t=0} = \mathcal{F}^{-1}\left(i\frac{d}{dt} e^{-ift} \hat{\varphi} \right)\Big|_{t=0}$

$= \mathcal{F}^{-1}(f \hat{\varphi}) = H_0 \varphi.$]

We are going to calculate an explicit formula for $e^{-iH_0 t}$:

Let $\psi \in \mathcal{S}(\mathbb{R}^3)$, then:

$$\psi_t(x) := \left(e^{-iH_0 t}\psi\right)(x)$$

$$= \left(\mathcal{F}^{-1} e^{-ift} \hat{\psi}\right)(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ip\cdot x} e^{-i\frac{p^2}{2}t} \hat{\psi}(p)\, dp.$$