THM 5.1: For $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $t > 0$:

$$\left(e^{-iH_0 t}\psi\right)(x) = e^{-i\frac{\pi}{4}n} \frac{1}{(2\pi |t|)^{n/2}} \int_{\mathbb{R}^n} e^{i(x-y)^2/2t} \psi(y)\, dy.$$

$\left(\text{for } t<0: e^{+i\frac{\pi}{4}n}\right)$

regularization provides integrability

PROOF: For $\psi \in \mathcal{S}(\mathbb{R}^n)$:

Then
$$\psi_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot x}\, e^{-\frac{p^2}{2}(\varepsilon+it)}\, \hat{\psi}(p)\, dp$$

$\left(\text{with the Gaussian } G_\varepsilon(p) = e^{-\frac{p^2}{2}(\varepsilon+it)}\right)$

$$= \lim_{\varepsilon \downarrow 0} \mathcal{F}^{-1}\left(G_\varepsilon \cdot \hat{\psi}\right)(x)$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^{n/2}} \left(\check{G}_\varepsilon * \psi\right)(x) \qquad (*)$$

where $\check{G}_\varepsilon(x) = \left(\frac{1}{\sqrt{\varepsilon+it}}\right)^n e^{-\frac{x^2}{2(\varepsilon+it)}}$ (by completing the square)

Here the complex square root is def. by main branch of complex logarithm

So $\sqrt{\varepsilon+it} \xrightarrow[(\varepsilon\downarrow 0)]{} \sqrt{|t|}\, e^{\mp i\pi/4}$.

Write out the convolution to obtain the desired formula.

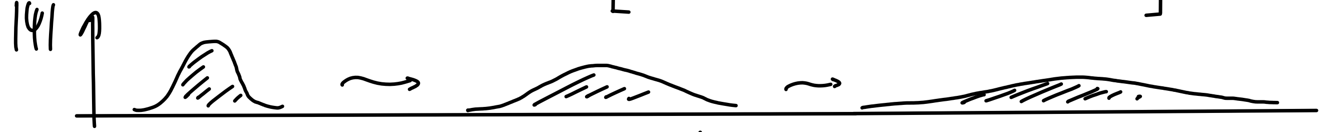
For $\psi \in L^1 \cap L^2$, formula $(*)$ is verified by approximating $\psi_k \to \psi$, $\psi_k \in \mathcal{S}(\mathbb{R}^n)$. ∎

COR. 5.2: (Dispersivity of the free evolution)

(i) For $\psi \in L^1 \cap L^2(\mathbb{R}^n)$: $\quad |\psi_t(x)| \leq \frac{1}{(2\pi|t|)^{n/2}} \|\psi\|_{L^1}$

(ii) For $\psi \in L^2(\mathbb{R}^n)$: $\quad \psi_t \longrightarrow 0 \;(t \to \infty)$.

[weak convergence, i.e. $\forall \varphi \in L^2 : \langle \varphi, \psi_t \rangle \to 0$]



$\longrightarrow$ increasing time t.

RMK: L^∞-norm: $\to 0 \;(t\to\infty)$, L^2-norm conserved (unitarity!).

PROOF: (i) $|\psi_t(x)| = \left| \frac{1}{(2\pi|t|)^{n/2}} \int e^{i(x-y)^2/2t} \psi(y)\, dy \right| \leq \frac{1}{(2\pi|t|)^{n/2}} \int |\psi(y)|\, dy.$

(ii) For $\varphi, \psi \in L^1 \cap L^2$ trivial.

For φ, ψ only in L^2:
Regularize $\varphi_N := \chi_{B_N(0)} \varphi$, $\psi_N = \chi_{B_N(0)} \psi$ and use $\varepsilon/3$-argument:

Let $\varepsilon > 0$. Let n so large that $\|\varphi_n - \varphi\| < \varepsilon/3$, $\|\psi_n - \psi\| < \varepsilon/3$.

Since $\varphi_n, \psi_n \in L^1 \cap L^2$:

$\exists T$ s.th. $t > T \;\Rightarrow\; |\langle \varphi_n, \psi_{n,t} \rangle| < \varepsilon/3$.

Thus $|\langle \varphi, \psi_t \rangle| \leq |\langle \varphi - \varphi_n, \psi_t \rangle| + |\langle \varphi_n, \psi_t - \psi_{n,t} \rangle| + |\langle \varphi_n, \psi_{n,t} \rangle|$

$$\leq \|\varphi - \varphi_n\| \, \|\psi\| + \overbrace{\|\varphi\|}^{\text{w.l.o.g.} \leq 1} \, \|\psi - \psi_n\| + |\langle \varphi_n, \psi_{n,t} \rangle|$$

$$< \quad \varepsilon/3 \quad + \quad \varepsilon/3 \quad + \quad \varepsilon/3. \qquad \blacksquare$$

DEF: Let $H = H^*$ in $\mathcal{H} = L^2(\mathbb{R}^n)$. (e.g. $H = H_0 + V$)

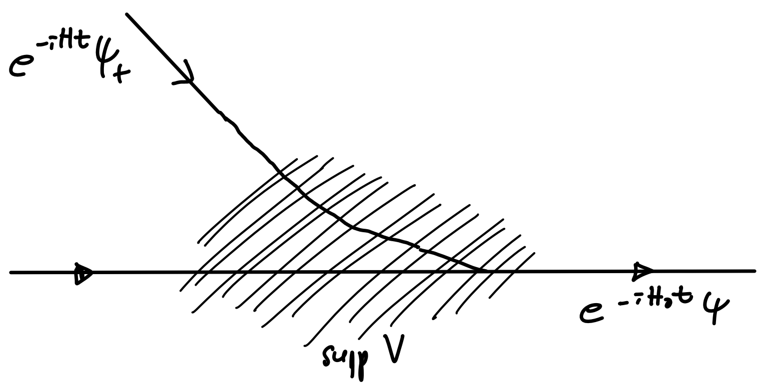
$\psi^+ \in \mathcal{H}$ is an <u>outgoing scattering state</u> if $\exists \psi \in \mathcal{H}$ such that

$$\| \underbrace{e^{-iHt}}_{\text{full evolution}} \psi^+ - \underbrace{e^{-iH_0 t}}_{\text{free evolution}} \psi \| \longrightarrow 0 \quad (t \to +\infty).$$

In this case, we call ψ^+ <u>asymptotically free</u> and $e^{-iH_0 t}\psi$ its <u>asymptotic state</u>.

Equivalently: $\psi^+ = \lim_{t \to +\infty} e^{iHt} e^{-iH_0 t} \psi$.

WARNING: Reed & Simon use ψ^+ for $t \to -\infty$!



<u>Ingoing scatt. state:</u> $\psi^- = \lim_{t \to -\infty} e^{iHt} e^{-iH_0 t} \psi$.

DEF: $V \in L^2_{loc}(\mathbb{R}^n) \;:\Longleftrightarrow\; \chi_K V \in L^2(\mathbb{R}^n)$ for all compact sets $K \subset \mathbb{R}^n$.

(While L^2 is also a condition on decay at infinity, L^2_{loc} only serves to prevent too strong singularities.)

PRP. 5.3: Let $H = H_0 + V$, $V \in L^2_{loc}(\mathbb{R}^3)$, V real-valued.

Assume there are $R > 0$, $\varepsilon > 0$, $c > 0$ such that

$$|V(x)| \leq \frac{c}{|x|^{1+\varepsilon}} \qquad \forall |x| > R.$$

Coulomb excluded! Requires modif. of wave operators due to its long range!

Then the wave operators $\Omega_\pm := \text{s-}\lim_{t \to \pm\infty} e^{iHt} e^{-iH_0 t}$ exist.

PROOF: Since $\|e^{iHt} e^{-iH_0 t}\| = 1$, it's enough to show that $\Omega_\pm \varphi$ exits for $\varphi \in \mathcal{S}(\mathbb{R}^3)$ (by Lem. 4.1).

Let $\varphi(t) := e^{iHt} e^{-iH_0 t} \varphi$.

We use Cook's argument:

always a good idea for comparing SCUGs.

Write $\varphi(t) = \varphi + \int_0^t \varphi'(s)\, ds$.

So $\lim_{t \to \infty} \varphi(t)$ exists if and only if $\int_0^\infty \varphi'(s)\, ds$ is convergent at $+\infty$.

Sufficient to show: $\int_R^\infty \|\varphi'(s)\|\, ds < \infty$. $\quad (*)$

arbitrary fixed lower bound, can be chosen conveniently.

By product rule:

$$\varphi'(s) = e^{iHs}\, i(H - H_0) e^{-iH_0 s} \varphi = e^{iHs}\, i V e^{-iH_0 s} \varphi.$$

(Think about domains! Why can I just differentiate?)

By unitarity of e^{iHs}: have to show $\int_R^\infty \|V e^{-iH_0 s} \varphi\|\, ds < \infty$.

[If $V \in L^2(\mathbb{R}^3)$: $\|V e^{-iH_0 s} \varphi\| \leq \|V\|_{L^2} \|e^{-iH_0 s} \varphi\|_{L^\infty}$

$$\leq \|V\|_{L^2} \|\varphi\|_{L^1} \underbrace{\frac{1}{(2\pi |s|)^{3/2}}}_{\text{integrable!}}.$$

]

This is not just an approx. arg. — we need a good idea!

For $V \notin L^2$: Split $V = V_2^t + V_\infty^t$ time-dependently:

$$V_\infty^t := V \chi_{\mathbb{R}^3 \setminus B_t(0)}, \qquad V_2^t := V \chi_{B_t(0)}.$$

Then:

- $$\| V_\infty^t \|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^3} \left| V(x) \chi_{\mathbb{R}^3 \setminus B_t(0)}(x) \right| \leq \frac{C}{|x|^{1+\varepsilon}} \chi_{\mathbb{R}^3 \setminus B_t(0)}(x) \underset{|x| \geq |t|}{\leq} \frac{C}{|t|^{1+\varepsilon}}$$

- $$\| V_2^t \|_{L^2} = \left(\int |V_2^t|^2 \chi_{B_t(0)}(x)\, d^3x \right)^{1/2} \leq \left(\int_{B_R(0)} |V|^2 \, d^3x + \int_{B_t(0) \setminus B_R(0)} \frac{C^2}{|x|^{2(1+\varepsilon)}} \, d^3x \right)^{1/2}$$

spherical coordinates

$$= \left(\text{const.} + \text{const.} \left(t^{1-2\varepsilon} - R^{1-2\varepsilon} \right) \right)^{1/2}$$

value of const. may change from line to line (but indep. of time t).

$$\leq \text{const.} \cdot \max\{1, t^{1/2-\varepsilon}\}.$$

Using these estimates:

$$\| V e^{-iH_0 s} \varphi \|_{L^2} \leq \| V_2^t e^{-iH_0 s} \varphi + V_\infty^t e^{-iH_0 s} \varphi \|_{L^2}$$

$$\underset{\text{Hölder}}{\leq} \| V_2^t \|_{L^2} \| e^{-iH_0 s} \varphi \|_{L^\infty} + \| V_\infty^t \|_{L^\infty} \| e^{-iH_0 s} \varphi \|_{L^2}$$

$$\leq \text{const.} \max\{1, t^{1/2-\varepsilon}\} \frac{1}{|t|^{3/2}} + \frac{C}{|t|^{1+\varepsilon}}$$

which is integrable at $t \to +\infty$. ∎

THM 5.4: Let $H = H^*$ and $H_0 = -\Delta/2$ in $L^2(\mathbb{R}^n)$.
If $\Omega_\pm$ exist, then:

(a) $\Omega_\pm : \mathcal{H} \to \mathcal{H}$ is an <u>isometry</u>, i.e. $\|\Omega_\pm \varphi\| = \|\varphi\|$ $\forall \varphi \in \mathcal{H}$.
(but not necessarily surjective.)

(b) <u>Intertwining:</u> $\Omega_\pm e^{-iH_0 t} = e^{-iHt} \Omega_\pm$ $\forall t \in \mathbb{R}$.
↳ Very useful to convert the evolution e^{-iHt} into the free evolution, which we understand very well.

(c) $\Omega_\pm H_0 \subset H \Omega_\pm$ (infinitesimal version of (b)).

(d) spectrum: $\sigma(H_0) \subset \sigma(H)$.

(e) $H\varphi = E\varphi \implies \varphi \perp \operatorname{range} \Omega_\pm$.
($\rightsquigarrow$ eigenstates are <u>not</u> asymptotically free; they are <u>bound</u> states!) , !

PROOF: (a) $\|\Omega_\pm \varphi\| = \lim_{t \to \pm\infty} \|e^{iHt} e^{-iH_0 t} \varphi\| = \|\varphi\|$.

(b) $\Omega_\pm e^{-iH_0 t} \varphi = \lim_{s \to \pm\infty} e^{iHs} e^{-iH_0 s} e^{-iH_0 t} \varphi$

$$= \lim_{s \to \pm\infty} e^{iHs} e^{-iH_0(s+t)} \varphi \overset{s' = s+t}{=} \lim_{s' \to \pm\infty} e^{iH(s'-t)} e^{-iH_0 s'} \varphi = e^{-iHt} \Omega_\pm \varphi.$$

(c) For $\varphi \in D(H_0)$: $\frac{i}{\varepsilon}(e^{-iH\varepsilon} - 1)\Omega_\pm \varphi = \Omega_\pm \frac{i}{\varepsilon}(e^{-iH_0\varepsilon} - 1)\varphi \xrightarrow[(\varepsilon \to 0)]{} \Omega_\pm H_0 \varphi$.

$\implies \Omega_\pm \varphi \in D(H)$ and $H\Omega_\pm \varphi = \Omega_\pm H_0 \varphi$.

(d) Apply intertwining: Let $\lambda \in \rho(H) \cap \mathbb{R}$. $\forall \varphi \in D(H)$:

$$\|(H_0 - \lambda)\varphi\| = \|\Omega_\pm (H_0 - \lambda)\varphi\| = \|(H - \lambda)\Omega_\pm \varphi\| \geq c\,\|\Omega_\pm \varphi\| = c\,\|\varphi\|.$$

$c^{-1} := \|(H-\lambda)^{-1}\|$

By same argument used already several times: $H_0 - \lambda$ invertible, $\lambda \in \rho(H_0)$.

[We have $\ker(H_0 - \lambda) = \{0\}$. Then $\operatorname{ran}(H_0 - \lambda)^\perp = \ker(H_0 - \lambda) = \{0\}$.
$\implies \overline{\operatorname{ran}(H_0 - \lambda)} = \mathcal{H}$. $\implies$ by closedness of H_0 and the Cauchy criterion: $\operatorname{ran}(H_0 - \lambda) = \mathcal{H}$. Boundedness for free.]

(e) Let $H\varphi = E\varphi$, $\psi \in \mathcal{H}$. Then

$$\langle \varphi, \Omega_\pm \psi \rangle = \lim_{t \to \pm\infty} \langle \varphi, e^{iHt} e^{-iH_0 t} \psi \rangle$$

$$= \lim_{t \to \pm\infty} e^{iEt} \langle \varphi, e^{-iH_0 t} \psi \rangle = 0$$

$e^{-iHt}\varphi = e^{-iEt}\varphi$
(verify this by taking the derivative)

By dispersivity:
Cor. 5.2: $e^{-iH_0 t}\psi \rightharpoonup 0.$

∎