QUADRATIC FORMS

1. Introduction

In this part we will introduce the notion of (unbounded) quadratic forms. This provides us with a convenient way to define self-adjoint operators from the energy, under suitable assumptions.

Three main results are to be remembered:

(1) Theorem 4 on closed and semi-bounded quadratic forms,
(2) the Friedrich extension of a positive symmetric operator,
(3) and the KLMN theorem, which may be seen as the quadratic form version of the Kato-Rellich theorem.

We emphasize the following important applications: the definition of the *magnetic Schrödinger operator* $|-i\nabla + \boldsymbol{A}|^2$, for the magnetic potential $\boldsymbol{A} \in L^2_{\text{loc}}(\mathbb{R}^3)^3$, and that of (minus) the Dirichlet Laplacian $-\Delta_D$, as the Friedrich extension of $-\Delta$ restricted to smooth functions in $C_0^\infty(\Omega)$ of an open domain $\Omega \subset \mathbb{R}^d$.

The proofs of the main results are interesting because they led us to consider different Hilbert spaces, one (continuously) embedded in another like

$$\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1} = \mathcal{H}'_{+1}.$$

For all of them Riesz lemma holds, enabling us to identify any of them with their continuous dual. Nevertheless we will still make the distinction between $\mathcal{H}_{+1}$, which is a subset of $\mathcal{H}$ and its dual $\mathcal{H}'_{+1}$ which contains $\mathcal{H}$. You have already encountered such a situation with Sobolev spaces:

$$H^n(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset H^{-n}(\mathbb{R}^d) = \big(H^n(\mathbb{R}^d)\big)'.$$

2. Quadratic forms

2.1. **Definition.** We start with the definition of the main object under consideration. As usual $\mathcal{H}$ denotes the underlying Hilbert space.

Definition 1. *A (densely defined) sesquilinear form is a map $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$ where*

(1) *the* form domain *$\mathcal{Q}(q) \subset \mathcal{H}$ is dense in $\mathcal{H}$,*
(2) *the map $\phi \mapsto q(\phi, \psi)$ is conjugate linear and the map $\phi \mapsto q(\psi, \phi)$ is linear.*

The quadratic form q associated to the sesquilinear map is the map:

$$q : \psi \in \mathcal{Q}(q) \mapsto q(\psi, \psi).$$

Remark 1. *Some authors do not make the distinction between sesquilinear forms and quadratic forms and call indifferently a quadratic form both functions $q(\cdot, \cdot)$ and $q(\cdot)$.*

We have indeed a one-to-one mapping between them and from a quadratic form $q(\cdot)$ we recover the underlying sesquilinear form by polarization[1]*:*

$$q(\phi,\psi)=\frac{1}{4}\Big[q(\phi+\psi)-q(\phi-\psi)+\frac{1}{i}(q(\phi+i\psi)-q(\phi-i\psi))\Big].$$

Definition 2. *Let q be a quadratic form.*

We say that q is symmetric if for all $\phi,\psi\in\mathcal{Q}(q)$ there holds: $q(\phi,\psi)=\overline{q(\psi,\phi)}$.

We say that q is semi-bounded from below if there exists $c\in\mathbb{R}$ such that for all $\psi\in\mathcal{Q}(q)$ there holds: $q(\psi,\psi)\geq c\|\psi\|_{\mathcal{H}}^2$. The number c is called a bound of the quadratic form.

Remark 2. *1. We will often say semi-bounded instead of bounded from below.*

2. Note that this definition extends to symmetric operators A: we say that such an operator is bounde from below if there exists $c\in\mathbb{R}$ such that for all $\psi\in\mathrm{dom}(A)$ we have $\langle\psi,A\psi\rangle\geq c\|\psi\|_{\mathcal{H}}^2$. The two notions are related as we will see with the Friedrich extension.

3. As $\mathcal{H}$ is a complex Hilbert space, if q is semi-bounded then it is automatically symmetric as we can check by developping the real *numbers $q(\phi+\lambda\psi)$ for $\lambda=1$ and $\lambda=i$.*

2.2. **First examples.**

2.2.1. *Quadratic form associated to a self-adjoint.* We give as first example the quadratic form q_A associated to a self-adjoint operator A. The form domain is:

$$\mathcal{Q}(q_A)=\mathcal{Q}(A):=\mathrm{dom}(|A|^{1/2})=\big\{\psi\in\mathcal{H},\ \langle\psi,|A|\psi\rangle=\int|x|\mathrm{d}\mu_\psi(x)<+\infty\big\},$$

where μ_ψ denotes the spectral measure associated to A and ψ. The quadratic form is given by the expectation of A:

$$q_A(\psi)=\langle\psi,A\psi\rangle=\int x\mathrm{d}\mu_\psi(x).$$

2.2.2. *Evaluation.* A second example is the evaluation function, say at 0 on smooth functions in $\mathcal{S}(\mathbb{R})\subset L^2(\mathbb{R})$:

$$\mathrm{ev}_0(f,g):=\overline{f(0)}g(0),\ f,g\in\mathcal{S}(\mathbb{R}).$$

It is not so well-behaved compared to the q. form coming from a semi-bounded s.a. operator.

2.3. **Closed quadratic forms and Freidrich extension.**

[1]A quadratic form is an algebraic notion. Here the field $\mathbb{C}$ has characteristic zero, hence there is no issue to divide by 4.

2.4. **Closed q. form.** Given a semi-bounded q. form q with bound c, we can define a new inner product $\langle\cdot,\cdot\rangle_q$ defined as follows:

$$\forall\, \phi,\psi \in \mathcal{Q}(q),\ \langle \phi,\psi\rangle_q := q(\phi,\psi) + (c+1)\langle\phi,\psi\rangle.$$

If we pick another bound c for the definition of the inner product we obviously obtain another inner product with *equivalent* norm. The space $(\mathcal{Q}(q),\langle\cdot,\cdot\rangle_q)$ is (pre)-Hilbert space, and we call $\mathcal{H}_q$ its closure under the norm $\|\psi\|_q := \sqrt{\langle\psi,\psi\rangle_q}$.

Observe that $\|\psi\|_q \geq \|\psi\|_{\mathcal{H}}$.

Definition 3. *A semi-bounded q. form q is said to be closed if $\mathcal{H}_q = \mathcal{Q}(q)$. Any subset $D \subset \mathcal{Q}(q)$ which is $\|\cdot\|_q$-dense is said to be a* form core *for q.*

It is said to be closable if $\mathcal{H}_q$ is a subset of $\mathcal{H}$.

Remark 3. *1. Let us be more clear on the closability. Consider the identity map:*

$$\iota : \begin{array}{ccc} (\mathcal{Q}(q),\|\cdot\|_q) & \longrightarrow & (\mathcal{H},\|\cdot\|_{\mathcal{H}}), \\ \psi & \mapsto & \psi, \end{array}$$

which is continuous as a map between the two above Banach spaces with norm smaller than 1. Hence it can be uniquely extended to a continuous map $\hat{\iota} : \mathcal{H}_q \to \mathcal{H}$ and we say that q is closable if $\hat{\iota}$ is injective.

*The evaluation ev_0 is **not** closed as we can see that $L^2(\mathbb{R}) \simeq \mathcal{H} \oplus \mathbb{C}$. However the quadratic form coming from a semi-bounded s.a. operator is closed (see second point below and check yourself!)*

2. To check that q is closed we have to see that a $\|\cdot\|_q$-Cauchy sequence (ψ_n) converges to some $\psi \in \mathcal{Q}(q)$ in the norm $\|\cdot\|_q$. As this norm controls $\|\cdot\|_{\mathcal{H}}$, then the sequence is $\|\cdot\|_{\mathcal{H}}$-Cauchy henceforth converges to some $\psi \in \mathcal{H}$ in the norm $\|\cdot\|_{\mathcal{H}}$.

Thus we obtain that q is closed iff *the following holds:*

***if** a sequence (ψ_n) in $\mathcal{Q}(q)$ satisfies $\|\psi_n - \psi\|_{\mathcal{H}} \to 0$ and $q(\psi_n - \psi_m) \underset{n,m\to\infty}{\longrightarrow} 0$, **then** $\psi \in \mathcal{Q}(q)$ and $q(\psi_n - \psi) \underset{n}{\longrightarrow} 0$.*

2.5. **Theorems.**

Theorem 4. *If q is semi-bounded and closed, then q corresponds to a unique s.a. operator A, which can be defined as follows:*

$$\begin{cases} \mathrm{dom}(A) := \big\{\psi \in \mathcal{Q}(q),\ \exists \widetilde{\psi} \in \mathcal{H},\ \forall \phi \in \mathcal{Q}(q),\ q(\phi,\psi) = \langle \phi, \widetilde{\psi}\rangle\big\}, \\ A\psi := \widetilde{\psi}. \end{cases} \tag{1}$$

"Conversely" if we consider the q. form associated to a positive[2] symmetric operator, then we can close it. This gives rise to the Friedrich extension.

Theorem 5. *[Friedrich extension] Let A be a positive sym. op. and q the q. form $q(\phi,\psi) := \langle\phi, A\psi\rangle$, $\phi,\psi \in \mathrm{dom}(A)$. Then*

(1) *q is closable with closure $\hat{q}$,*

[2]or semi-bounded. Indeed up to shifting by the bound: $A \to A - c$ we can assume that A is positive.

(2) *$\hat{q}$ is the q. form of a unique s.a. op. $\hat{A}$,*

(3) *$\hat{A}$ is a pos. extension of A and* $\inf \sigma(\hat{A}) = \inf_{\psi\in \mathrm{dom}(A)\setminus\{0\}} \frac{\langle \psi, A\psi\rangle}{\langle \psi,\psi\rangle}$.

(4) *$\hat{A}$ is the unique s.a. extension of A whose domain is in $\mathcal{Q}(\hat{q})$.*

2.6. **Proofs.**

2.6.1. *Proof of Thm 4.* Observe that up to replacing q by $q-c$, we can assume w.l.o.g. that the bound of q is 0, that is q positive.

As $\mathcal{Q}(q)$ is dense in $\mathcal{H}$, then A in (1) is well-defined by Riesz lemma (if such a $\widetilde{\psi}$ exists then it is unique). Then it is clear that 0 is in $\mathrm{dom}(A)$.

A is pos. and sym. Let $\phi, \psi \in \mathrm{dom}(A)$. By construction we have:

$$\langle \psi, A\psi\rangle = q(\psi) \geq 0,$$

hence A is pos. Similarly we have:

$$\begin{aligned}\langle \phi, A\psi\rangle &\overset{def}{=} q(\phi,\psi),\\ &\overset{q\text{ sym.}}{=} \overline{q(\psi,\phi)},\\ &\overset{def}{=} \overline{\langle \psi, A\phi\rangle},\\ &= \langle A\phi, \psi\rangle.\end{aligned}$$

A is s.a. We take a detour and show that $(1+A)^{-1}$ is well-defined, everywhere defined bounded and symmetric. In other words, $(1+A)^{-1}$ is bounded and self-adjoint. This implies $1+A$ s.a. hence A s.a., we can use for instance the multiplication form of the spectral theorem and check it on $L^2(\mathbb{R}, \mathrm{d}\mu)$ when $(1+A)^{-1}$ corresponds to the multiplication by x.

Claim: $\mathrm{ran}(1+A) = \mathcal{H}$ This is an application of Riesz lemma in the Hilbert space $(\mathcal{Q}(q), \|\cdot\|_q)$. Indeed, given $\psi \in \mathcal{H}$, the following linear form is bounded:

$$\phi \in \mathcal{Q}(q) \mapsto \langle \psi, \phi\rangle.$$

By Riesz lemma, there exists $\widetilde{\psi} \in \mathcal{Q}(q)$ such that for all $\phi \in \mathcal{Q}(q)$ we have:

$$\langle \psi, \phi\rangle = \langle \widetilde{\psi}, \phi\rangle_q = q(\widetilde{\psi}, \phi) + \langle \widetilde{\psi}, \phi\rangle.$$

By definition of A, we get $\widetilde{\psi} \in \mathrm{dom}(A)$ and $\psi = (1+A)\widetilde{\psi}$.

Let us now check that $1+A$ is injective: it simply follows from positivity of A and Cauchy-Schwarz inequality. Indeed for $\psi \in \mathrm{dom}(A)$, we have:

$$\|(1+A)\psi\|_{\mathcal{H}} \geq \langle (1+A)\psi, \psi\rangle \geq \|\psi\|_{\mathcal{H}}^2.$$

Hence $(1+A)^{-1} : (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \to (\mathrm{dom}(A), \|\cdot\|_{\mathcal{H}})$ is well-defined (and continuous with norm smaller than 1). At last we check that $(1+A)^{-1}$ is symmetric. Given $\widetilde{\psi}_1, \widetilde{\psi}_2 \in \mathcal{H}$ we know that there exists uniques $\psi_1, \psi_2 \in \mathrm{dom}(A)$ with

$$\widetilde{\psi}_j = (1+A)\psi_j.$$

As $\mathcal{Q}(q)$ is dense in $\mathcal{H}$ and $(1+A)^{-1}$, we can assume w.l.o.g. that $\psi_j \in \mathcal{Q}(q)$. A computation yields:

$$\begin{aligned}\langle (1+A)^{-1}\widetilde{\psi}_1, \widetilde{\psi}_2\rangle &= \langle \psi_1, (1+A)\psi_2\rangle,\\ &\stackrel{def}{=} \langle \psi_1,\psi_2\rangle + q(\psi_1,\psi_2),\\ &= \overline{\langle \psi_2,\psi_1\rangle + q(\psi_2,\psi_1)},\\ &\stackrel{def}{=} \overline{\langle \psi_2,(1+A)\psi_1\rangle},\\ &= \langle \widetilde{\psi}_1, (1+A)^{-1}\widetilde{\psi}_2\rangle.\end{aligned}$$

A is unique. Let $\widetilde{A}$ be another s.a. op. associated to q, in particular we have $\mathrm{dom}(\widetilde{A}) \subset \mathcal{Q}(q)$ and for all $\psi \in \mathrm{dom}(\widetilde{A})$ and $\phi \in \mathcal{Q}(q)$ there holds $\langle \phi, \widetilde{A}\psi\rangle = q(\phi,\psi)$. Now if we assume $\phi \in \mathrm{dom}(A) \subset \mathcal{Q}(q)$, we obtain:

$$\langle \phi, \widetilde{A}\psi\rangle = q(\phi,\psi) = \langle A\phi,\psi\rangle,$$

hence $\widetilde{A}$ extends A. As they are both s.a. they are equal.

2.6.2. *Proof of Thm 5.*
q is closable. We show that q is closable. As said in Remark 3, we check that the map $\hat{\imath}$ is injective. Let $\psi \in \mathcal{H}_q$ (the closure under $\|\cdot\|_q$) with $\hat{\imath}(\psi) = 0$. This means that there exists a sequence $(\psi_n)_n$ in $\mathrm{dom}(A)$ which is a $\|\cdot\|_q$-Cauchy sequence and for which there holds $\|\psi_n\|_{\mathcal{H}} \to 0$. In particular $(\|\psi_n\|_q)_n$ is bounded by $M < +infty$.

Let us show $\|\psi_n\|_q^2 \to 0$. For $n, m \geq 1$, we have:

$$\begin{aligned}\|\psi_n\|_q^2 &= \langle \psi_n,\psi_n\rangle_q \quad (= \langle \psi_n, (1+A)\psi_n\rangle),\\ &= \langle \psi_n, \psi_n - \psi_m\rangle_q + \langle \psi_n,\psi_m\rangle_q,\\ &= \langle \psi_n, \psi_n - \psi_m\rangle_q + \langle (1+A)\psi_n,\psi_m\rangle\\ &\leq \|\psi_n\|_q\|\psi_n-\psi_m\|_q + \|(1+A)\psi_n\|_{\mathcal{H}}\|\psi_m\|_{\mathcal{H}}.\end{aligned}$$

We first take the liminf in m, and get:

$$\|\psi_n\|_q^2 \leq \|\psi_n\|_q \liminf_{m\to+\infty}\|\psi_n - \psi_m\|_q + 0.$$

As $(\psi_n)_n$ is Cauchy for $\|\cdot\|_q$, by taking the limit $n \to +\infty$ we obtain $\lim_{n\to+\infty}\|\psi_n\|_q^2 \leq M \times 0 + 0 = 0$.

So q is closable and by Thm 4, its closure is associated to a unique $\hat{A}$.

$\hat{A}$ extends A. For $\phi \in \mathrm{dom}(A)$ and $\psi \in \mathrm{dom}(\hat{A})$, we have:

$$\langle A\phi,\psi\rangle = \hat{q}[\phi,\psi] = \langle \phi, \hat{A}\psi\rangle, \tag{2}$$

hence $\hat{A}$ extends A.

Uniqueness. Let A_1 be a symmetric extension of A with $\mathrm{dom}(A_1) \subset \mathcal{Q}(\hat{q})$. By replacing A by A_1 in (2) we get that $\hat{A}$ extends A_1. So if A_1 is self-adjoint then they are equal.

Bottom of the spectrum. The formula simply follows from the fact that $\mathrm{dom}(A)$ is $\|\cdot\|_q$-dense in $\mathcal{Q}(\hat{q})$.

3. The KLMN theorem

3.1. **Statement of the theorem.** It is named after Kato, Lions, Lax, Milgram and Nelson.

Theorem 6. *Let A be a positive s.a. operator and $\beta(\psi, \psi)$ a symmetric q. form on $\mathcal{Q}(A)$ such that there exists $0 < a < 1$ and $b \in \mathbb{R}$ such that for all $\psi \in \operatorname{dom}(A)$ there holds:*

$$|\beta(\phi, \phi)| \leq q\langle\phi, A\phi\rangle + b\|\phi\|_{\mathcal{H}}^2.$$

Then the following holds.

(1) *There exists a unique s.a. op C with $\mathcal{Q}(C) = \mathcal{Q}(A)$ and for all ψ, ϕ in the common form domain we have:*

$$q_C(\phi, \psi) = q_A(\phi, \psi) + \beta(\phi, \psi).$$

(2) *C is bounded from below by $-b$ and any domain of essential self-adjointness of A is a form core for q_C.*

Proof. Define $\gamma(\phi, \psi) := q_A(\phi, \psi) + \beta(\phi, \psi)$ on $\mathcal{Q}(A)$. We have:

$$\gamma(\phi, \phi) \geq (1 - a)q_A(\phi, \phi) - b\|\phi\|_{\mathcal{H}}^2,$$

hence γ is bounde from below by $-b$. We claim that γ is already closed. it suffices to show that $\|\cdot\|_{q_A}$ and $\|\cdot\|_\gamma$ are equivalent norms on $\mathcal{Q}(A)$. As $\mathcal{Q}(A)$ is $\|\cdot\|_{q_A}$-closed, then it will also be $\|\cdot\|_\gamma$-closed.

We have:

$$(1-a)q_A(\phi, \phi)+(2|b|-b)\langle\phi, \phi\rangle \leq \gamma(\phi, \phi)+2|b|\langle\phi, \phi\rangle \leq (1+a)q_A(\phi, \phi)+(1+2|b|)\langle\phi, \phi\rangle,$$

which proves the equivalence of the two norms on $\mathcal{Q}(A)$.

The statement about domain of self-adjointness follows from the fact that the graph norm of A on $\operatorname{dom}(A)$ controls $\|\cdot\|_{q_A}$:

$$\langle\psi, (1 + A)\psi\rangle \leq \langle\psi, \psi\rangle + 2^{-1}(\|\psi\|_{\mathcal{H}}^2 + \|A\psi\|_{\mathcal{H}}^2) \leq 3/2\|\psi\|_A^2.$$

As a domain of self-adjointness is dense (in $\operatorname{dom}(A)$) under the graph norm, it is dense under $\|\cdot\|_{q_A}$ in $\mathcal{Q}(A)$ (because so is $\operatorname{dom}(A)$). □

3.2. **Relative form bound.** The KLMN theorem leads us to the following definition.

Definition 4. *Let A be a positive s.a. op and B is a s.a. operator.*

B is said to be relatively form bounded w.r.t. A with relative bound $a > 0$ if $\mathcal{Q}(A) \subset \mathcal{Q}(B)$ and if there exists $b \in \mathbb{R}$ such that:

$$|q_B(\phi, \phi)| \leq aq_A(\phi, \phi) + b\|\phi\|_{\mathcal{H}}^2.$$

If for any $a > 0$, B is relatively form bounded w.r.t. A with relative bound a, then B is said to be infinitesimally form bounded w.r.t. A and we write $B \lll A$.

Mutatis mutandis *a similar definition holds for a quadratic form β defined on $\mathcal{Q}(A)$.*

Remark 7. *It can be shown that if B is infinitesimally operator bounded w.r.t. A, then it is also infinitesimally form bounded w.r.t. A.*

We now give several examples.

3.3. **Examples.**

3.3.1. *Evaluation.* As a first example for the KLMN theorem, we consider the evaluation ev_0 introduced earlier. Itself it is not closable, but using the Fourier transform in dimension 1, it can be easily check that ev_0 is infinitesimally form bounded w.r.t. $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

3.3.2. *Homogeneous potentials.* We recall the Hardy's inequality on $L^2(\mathbb{R}^d)$ with $d \geq 3$ which states the following:

$$\forall \psi \in H^1(\mathbb{R}^d), \ \int \frac{|\psi(x)|^2}{|x|^2}\mathrm{d}x \leq \frac{4}{(d-2)^2}\int |\nabla\psi|^2 \tag{3}$$

which implies that $|\cdot|^{-2}$ is relatively form bounded w.r.t. $-\Delta_{\mathbb{R}^d}$ with relative bound $\frac{(d-2)^2}{4}$.

This implies that for all $0 < \alpha < 2$, there holds $-|\cdot|^{-\alpha} \lll -\Delta$. Indeed for all $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we have the following operator inequality:

$$|x|^{-\alpha} \leq |x|^{-\alpha}(\mathbb{1}_{(|x|\leq\varepsilon)} + \mathbb{1}_{(|x|>\varepsilon)}) \leq \varepsilon^{2-\alpha}|x|^{-2} + \varepsilon^{-\alpha}.$$

If you have never seen Hardy's inequality, here is one proof.

For $d \geq 3$, it suffices to establish it for smooth functions $\psi \in \mathbb{C}_0^\infty(\mathbb{R}^d)$. We will extend it to $H^1(\mathbb{R}^d)$ by density. For such a ψ, let $f := |\psi|^2$.

Observe that on $\mathbb{R}^d\backslash\{0\}$ the divergence of the vector field $\mathbf{V}(x) := f(x)\frac{x}{|x|^3}$ is:

$$[\nabla\cdot\mathbf{V}](x) = \langle\nabla f(x), \tfrac{x}{|x|^2}\rangle + f\big[\frac{d}{|x|^2} - \sum_j \frac{2x_j^2}{|x|^4}\big] = \langle\nabla f(x), \tfrac{x}{|x|^2}\rangle + (d-2)\frac{f(x)}{|x|^2}.$$

We write S_ε the hyper-sphere of radius ε and $\mathbb{S} = \mathbb{S}^{d-1}$ that of radius 1. By Stokes formula we obtain:

$$\begin{aligned}(d-2)\int_{|x|\geq\varepsilon}\frac{f(x)}{|x|^2}\mathrm{d}x &= \int_{|x|\geq\varepsilon}[\nabla\cdot\mathbf{V}](x)\mathrm{d}x - \int_{|x|\geq\varepsilon}\langle\nabla f(x), \tfrac{x}{|x|^2}\rangle\mathrm{d}x,\\ &= -\int_{y\in S_\varepsilon} f(y)\langle\tfrac{y}{|y|^2}, n_{S_\varepsilon}(y)\rangle\mathrm{d}_{S_\varepsilon}(y) - \int_{|x|\geq\varepsilon}\langle\nabla f(x), \tfrac{x}{|x|^2}\rangle\mathrm{d}x,\\ &= -\int_{n\in\mathbb{S}} f(\varepsilon n)\langle\tfrac{n}{\varepsilon}, n\rangle\varepsilon^2\mathrm{d}_{\mathbb{S}}(n) - \int_{|x|\geq\varepsilon}\langle\nabla f(x), \tfrac{x}{|x|^2}\rangle\mathrm{d}x.\end{aligned}$$

Taking the limit $\varepsilon \to 0$ yields:

$$\begin{aligned}(d-2)\int\frac{f(x)}{|x|^2}\mathrm{d}x &= -\int\langle\nabla f(x), \tfrac{x}{|x|^2}\rangle\mathrm{d}x,\\ &\leq 2\int|\nabla\psi(x)|\frac{|\psi(x)|}{|x|}\mathrm{d}x.\end{aligned}$$

By cauchy-Schwartz inequality, we obtain:

$$\Big((d-2)\int\frac{|\psi(x)|^2}{|x|^2}\mathrm{d}x\Big)^2 \leq 4\int|\nabla\psi(x)|^2\mathrm{d}x\int\frac{|\psi(x)|^2}{|x|^2}\mathrm{d}x,$$

from which we obtain Hardy's inequality. Observe in fact that we can refine it: following the proof we realize that we can replace $|\nabla\psi|$ by $|\nabla|\psi||$.

3.3.3. *Rollnik potentials.* In dimension $d = 3$, we introduce the Banach space $\mathcal{R}$ of Rollnik potentials.

$$\mathcal{R} := \left\{ V \text{ measurable, } \|V\|_{\mathcal{R}}^2 := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} \mathrm{d}x\mathrm{d}y < + < infty \right\}. \tag{4}$$

Let $V \in \mathcal{R}$, calling f_V the Fourier transform of $|V|$, we have:

$$\begin{aligned} \|V\|_{\mathcal{R}}^2 &= \int |V| \times |V| * \tfrac{1}{|\cdot|^2}, \\ &= \frac{1}{4\pi} \int \frac{|f_V(p)|^2}{|p|} \mathrm{d}p. \end{aligned}$$

Similarly, if we consider $V, W \in \mathcal{R}$, we obtain by Cauchy-Schwarz inequality:

$$\begin{aligned} \|V + W\|_{\mathcal{R}}^2 &\le \frac{1}{4\pi} \int \frac{(|f_V(p)| + |f_W(p)|)^2}{|p|} \mathrm{d}p, \\ &\le (\|V\|_{\mathcal{R}} + \|W\|_{\mathcal{R}})^2, \end{aligned}$$

which establishes the triangle inequality.

In this course, this class has to be understood as the set of potentials for which interesting results can be stated with few technicalities.

Remark 8 ($L^{3/2}(\mathbb{R}^3) \subset \mathcal{R}$)**.** *Thanks to a special case of the Hardy-Litllewood-Sobolev inequality* [1] *which states that the following linear map is continuous*

$$V \in L^{3/2}(\mathbb{R}^3) \mapsto V * \frac{1}{|\cdot|^2} \in L^3(\mathbb{R}^3),$$

we get $L^{3/2}(\mathbb{R}^3) \subset \mathcal{R}$*.*

We claim:

Lemma 9. *A (real valued) potential* $V \in \mathcal{R} + L^\infty(\mathbb{R}^3)$ *is infinitesimally form bounded w.r.t.* $-\Delta_{\mathbb{R}^3}$*.*

W.l.o.g. it suffices to check it for $V \in \mathcal{R}$. We will prove this lemma in a another part of the lecture dedicated to the study of Rollnik potentials.

We can then derive an analogue of Kato's theorem on atomic Schrödinger operators.

Theorem 10. *Let* $N \in \mathbb{N}$ *and let* V_i, V_{ij} *be (real-valued) potentials in* $\mathcal{R} + L^\infty(\mathbb{R}^3)$*,* $1 \le i, j \le N$*. Let*

$$V(x) := \sum_{i=1}^N V_i(x_i) + \sum_{1 \le i,j \le N} V_{ij}(x_i - x_j), \ x = (x_1, \dots, x_N) \in \mathbb{R}^{3N}.$$

Then $V \lll -\Delta_{\mathbb{R}^{3N}}$ *and* $-\Delta_{\mathbb{R}^{3N}} + V$ *is a well-defined s.a. op with domain* $H^2(\mathbb{R}^{3N})$*.*

References

[1] Elliott H. Lieb and Michael Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. MR1817225

[2] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.

[3] Gerald Teschl, *Mathematical methods in quantum mechanics*, 2nd ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. With applications to Schrödinger operators.