ROLLNIK POTENTIALS

1. Introduction

In this part we will study in more details the potentials V which are in the Rollnik class. We recall that it corresponds to the Banach space

$$\mathcal{R} := \Big\{V \text{ measurable in } \mathbb{R}^3,\ \|V\|_{\mathcal{R}}^2 = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y < +\infty\Big\},$$

and that by the Hardy-Littlewood-Sobolev inequality [1], we have:

$$L^{3/2}(\mathbb{R}^3) \subset \mathcal{R}.$$

So this class contains a more well known class of functions. In this course, they constitute a class of potentials for which interesting results can be proven with few technicalities.

Here, we aim to prove two results, which we gather in the following proposition.

Proposition 1. *Let $V \in \mathcal{R}$ real valued. Then*

1. *V is infinitesimally form bounded w.r.t. $-\Delta$.*
2. *There exists $a > 0$ such that $(-\Delta + V + a^2)^{-1} - (-\Delta + a^2)^{-1}$ is compact, and $\sigma_{\mathrm{ess}}(-\Delta + V) = \sigma_{\mathrm{ess}}(-\Delta) = [0, +\infty)$.*

The statement on the essential spectrum simply follows from the Weyl theorem we have seen on the stability of the essential spectrum: for two s.a. operator, if the difference of the resolvent is compact, then their essential spectra coincide.

Along the prooof we will use the important result of *operator monotonicity* of the inverse.

Lemma 2. *Let A, B two positive s.a. operators satisfying:*

$$0 \le A \le B.$$

If A is invertible, then so is B and we have:

$$0 \le B^{-1} \le A^{-1}.$$

Remark 3. *The operator inequality $0 \le A \le B$ has to be understood in the following sense: the inclusion $\mathcal{Q}(B) \subset \mathcal{Q}(A)$ holds, and for all $\psi \in \mathcal{Q}(B)$ we have:*

$$0 \le q_A[\psi,\psi] \le q_B[\psi,\psi],$$

where q_A and q_B denote the corresponding quadratic forms. Of course for $\psi \in \mathrm{dom}(B) \cap \mathrm{dom}(A)$ this means:

$$0 \le \langle\psi, A\psi\rangle \le \langle\psi, B\psi\rangle.$$

2. Proof of Proposition 1

Auxiliary result. We first prove an auxiliary result and show that for all $a > 0$, the following operators are compact:

$$K_a := |V|^{1/2}(-\Delta + a^2)^{-1}|V|^{1/2} = k_a^* k_a,\ k_a := (-\Delta + a^2)^{-1/2}|V|^{1/2},$$

and that we have: $\lim_{a\to+\infty}\|k_a\|_{\mathcal{L}} = 0$.

1. Observe that K_a has integral kernel:

$$K_a(x,y) = \frac{1}{4\pi}|V(x)|^{1/2}\frac{e^{-a|x-y|}}{|x-y|}|V(y)|^{1/2}.$$

Indeed, $|V|^{1/2}$ is a multiplication operator while we know that $(-\Delta + a^2)^{-1}$ acts by convolution by the Yukawa potential:

$$Y_a(x) = \frac{1}{4\pi}\frac{e^{-a|x|}}{|x|}.$$

Putting everything together we obtain:

$$(K_a\psi)(x) = \int_{y\in\mathbb{R}^3}\frac{1}{4\pi}|V(x)|^{1/2}\frac{e^{-a|x-y|}}{|x-y|}|V(y)|^{1/2}\psi(y)\mathrm{d}y.$$

2. This integral kernel $K_a(\cdot,\cdot)$ is $L^2(\mathbb{R}^3\times\mathbb{R}^3)$.

You are asked in the fourth assignment to show that this implies that K_a is compact. In fact it is even more: it is Hilbert-Schmidt, and writing $\lambda_1 \geq \lambda_2 \geq \cdots$ the sequence of eigenvalues of K_a counted with multiplicity we have:

$$\iint |K_a(x,y)|^2\mathrm{d}x\mathrm{d}y = \sum_{n\in\mathbb{N}}|\lambda_n|^2.$$

Here a simple computation gives:

$$\begin{aligned}\iint |K_a(x,y)|^2\mathrm{d}x\mathrm{d}y &= \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|V(x)|e^{-2a|x-y|}|V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y,\\ &\leq \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|V(x)||V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y < +\infty.\end{aligned}$$

3. We have $\lim\|K_a(\cdot,\cdot)\|_{L^2} = 0$. This follows from the dominated convergence. Indeed we have:

- $|K_a(x,y)|^2 \leq \frac{1}{(4\pi)^2}\frac{|V(x)||V(y)|}{|x-y|^2}$, latter function which is integrable in $\mathbb{R}^3\times\mathbb{R}^3$,
- $\lim_{a\to+\infty}|K_a(x,y)|^2 = 0$ for all $x \neq y$, hence almost everywhere in $\mathbb{R}^3\times\mathbb{R}^3$.

4. By the Cauchy-Schwarz inequality we have:

$$\int\left|\int K_a(x,y)\psi(y)\mathrm{d}y\right|^2\mathrm{d}x \leq \int\mathrm{d}x\int|K_a(x,y)|^2\mathrm{d}y\int|\psi(y')|^2\mathrm{d}y' \leq \|K_a(\cdot,\cdot)\|_{L^2}^2\|\psi\|_{L^2}^2.$$

Hence we recover $\|K_a\|_{\mathcal{L}} \leq \|K_a(\cdot,\cdot)\|_{L^2}$. Furthermore there holds:

$$\|K_a\|_{\mathcal{L}} = \|k_a^* k_a\|_{\mathcal{L}} = \|k_a\|_{\mathcal{L}}^2 \underset{a\to+\infty}{\to} 0.$$

5. k_a is compact. It is obvious, but let us check it. Let us pick a sequence

(ψ_n) in $L^2(\mathbb{R}^3)$ which converges weakly to ψ. We aim to show that $(k_a\psi_n)$ converges in norm. We have:

$$\|k_a(\psi_n-\psi)\|_{L^2}^2=\|k_a\psi_n\|_{L^2}+\|\psi\|_{L^2}^2+2\mathrm{Re}\langle k_a\psi_n,k_a\psi\rangle.$$

As k_a is bounded, we have $\mathrm{Re}\langle k_a\psi_n,k_a\psi\rangle\to\mathrm{Re}\langle k_a\psi,k_a\psi\rangle=\|k_a\psi\|_{L^2}^2$. Hence we have norm convergence if and only if we have convergence of the norm[1] $\|k_a\psi_n\|_{L^2}$. Here we have:

$$\|k_a\psi_n\|_{L^2}^2=\langle K_a\psi_n,\psi_n\rangle\to\langle K_a\psi,\psi\rangle=\|k_a\psi\|_{L^2}^2.$$

We have used the fact that $(K_a\psi_n)$ converges strongly.

Proof of $V\lll-\Delta$. It simply follows the fact that:

$$\varepsilon(a):=\|(-\Delta+a^2)^{-1/2}|V|(-\Delta+a^2)^{-1/2}\|_{\mathcal{L}}=\|k_ak_a^*\|_{\mathcal{L}}=\|k_a\|_{\mathcal{L}}^2\underset{a\to+\infty}{\longrightarrow}0.$$

Let $\psi\in H^1(\mathbb{R}^3)=\mathcal{Q}(-\Delta)$. We have:

$$\begin{aligned}|\langle\psi,V\psi\rangle|&\le\int|V||\psi|^2=\langle\psi,|V|\psi\rangle,\\&\le\left\langle(-\Delta+a^2)^{1/2}\psi,\ (-\Delta+a^2)^{-1/2}|V|(-\Delta+a^2)^{-1/2}(-\Delta+a^2)^{1/2}\psi\right\rangle,\\&\le\varepsilon(a)\|(-\Delta+a^2)^{1/2}\psi\|_{L^2}^2=\varepsilon(a)\|\nabla\psi\|_{L^2}^2+a^2\varepsilon(a)\|\psi\|_{L^2}^2.\end{aligned}$$

End of the proof. Up to taking $a>0$ big enough we have $0<\varepsilon(a)<2^{-1}$. Hence for $\psi\in H^2(\mathbb{R}^3)$, we get:

$$0\le2^{-1}\langle\psi,(-\Delta+a^2)\psi\rangle\le\langle\psi,(-\Delta+a^2+V)\psi\rangle\le3/2\langle\psi,(-\Delta+a^2)\psi\rangle.$$

This inequality naturally extends to $H^1(\mathbb{R}^3)$ by density (but we have to rewrite the inequalities in terms of the corresponding quadratic forms). By Lemma 2, we get:

$$2/3(-\Delta+a^2)^{-1}\le(-\Delta+a^2+V)^{-1}\le2(-\Delta+a^2)^{-1}.$$

By conjugating with $(-\Delta+a^2)^{1/2}$, we obtain:

$$2/3\le(-\Delta+a^2)^{1/2}(-\Delta+a^2+V)^{-1}(-\Delta+a^2)^{1/2}\le2,$$

hence $(-\Delta+a^2)^{1/2}(-\Delta+a^2+V)^{-1}(-\Delta+a^2)^{1/2}$ is a bounded self-adjoint operator.

Now consider the difference of the resolvents:

$$\begin{aligned}(-\Delta+a^2+V)^{-1}-(-\Delta+a^2)^{-1}&=(-\Delta+a^2)^{-1}V(-\Delta+a^2+V)^{-1},\\&=\big[(-\Delta+a^2)^{-1}|V|^{1/2}\big]\mathrm{sign}(V)\big[|V|^{1/2}(-\Delta+a^2)^{-1/2}\big]\\&\quad\times\big[(-\Delta+a^2)^{1/2}(-\Delta+a^2+V)^{-1}(-\Delta+a^2)^{1/2}\big](-\Delta+a^2)^{-1/2}.\end{aligned}\tag{1}$$

It is compact as the composition of compact and bounded operators.

[1] We say that there is no loss of mass.

3. Proof of Lemma 2

As A is invertible, then $0 \notin \sigma(A)$, and there exists $\varepsilon > 0$ with:

$$\varepsilon \leq A.$$

In particular for all $\psi \in \mathrm{dom}(B) \subset \mathcal{Q}(B) \subset \mathcal{Q}(A)$, we have:

$$\varepsilon \|\psi\|_{\mathcal{H}}^2 \leq q_A(\psi, \psi) \leq q_B(\psi, \psi) \leq \langle \psi, B\psi \rangle.$$

This shows that B is injective and that 0 is not in the spectrum of B (if you are not convinced you can argue by contradiction and by taking a Weyl sequence). Furthermore the inequality shows that B^{-1} is bounded, positive with norm smaller than ε^{-1}.

Let $\psi \in \mathcal{H}$. We show that $\langle \psi, B^{-1}\psi \rangle \leq \langle \psi, A^{-1}\psi \rangle$. We introduce $\phi \in \mathcal{Q}(A)$ to be chosen later. Using the positivity of A, we have:

$$\begin{aligned} 0 \leq q_A(\phi - A^{-1}\psi, \phi - A^{-1}\psi) &= q_A(\phi, \phi) - 2\mathrm{Re}\, q_A(\phi, A^{-1}\psi) + q_A(A^{-1}\psi, A^{-1}\psi), \\ &= q_A(\phi, \phi) - 2\mathrm{Re}\langle \phi, \psi \rangle + \langle \psi, A^{-1}\psi \rangle. \end{aligned}$$

Hence we have:

$$\begin{aligned} \langle \psi, A^{-1}\psi \rangle &\geq 2\mathrm{Re}\langle \phi, \psi \rangle - q_A(\phi, \phi), \\ &\geq 2\mathrm{Re}\langle \phi, \psi \rangle - q_B(\phi, \phi). \end{aligned}$$

Choosing $\phi = B^{-1}\psi \in \mathrm{dom}(B) \subset \mathcal{Q}(B) \subset \mathcal{Q}(A)$, we obtain:

$$\langle \psi, A^{-1}\psi \rangle \geq 2\langle B^{-1}\psi, \psi \rangle - \langle \psi, B^{-1}\psi \rangle = \langle \psi, B^{-1}\psi \rangle.$$

References

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[3] Gerald Teschl, *Mathematical methods in quantum mechanics*, 2nd ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. With applications to Schrödinger operators.