Jérémy Sok
Niels Benedikter

Advanced Mathematical Physics

Assignment 3 of 4

To be handed in on Friday, June 2, at the beginning of the seminar.

Problem 1: Defect indices (10 points)

Consider the operator

$$A = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad D(A) = C_0^\infty((0,\infty)).$$

Compute its defect indices. Discuss the existence of self-adjoint extensions of A.

Problem 2: Spectral calculus (5+5 points)

a. Let P be a self-adjoint operator acting in a Hilbert space $\mathcal{H}$.

Show that P is a projection if and only if $\sigma(P) \subset \{0,1\}$.

b. Consider the parity operator $\Pi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, acting by $(\Pi\psi)(x) = \psi(-x)$.

Show that Π is self-adjoint, compute its spectrum and find the associated projection-valued measure P_Π.

Problem 3: Wave Operators for Relativistic Quantum Mechanics (10 points)

We introduce a model for a spinless particle of mass $m = 1$, with speed of light $c = 1$ and relativistic dispersion relation. Consider the Hamiltonian

$$H = H_0 + V, \quad H_0 = \sqrt{-\Delta + 1} := \mathscr{F}^{-1} T_\omega \mathscr{F},$$

where $\omega(p) := \sqrt{p^2+1}$, and $V \in L^\infty(\mathbb{R}^n)$ is real-valued. You may take for granted that $H = H^*$ on $D(H_0)$. Assume the potential to satisfy

$$|V(x)| \leq \frac{\text{const}}{|x|^\mu}, \quad \text{for } |x| > R, \text{ and with } \mu > 1.$$

Prove existence of the wave operator $\Omega_+ = \text{s-lim}_{t\to\infty} e^{iHt}e^{-iH_0 t}$.

Hint 1: Consider $\varphi \in X \subset L^2(\mathbb{R}^n)$, where X is the dense subspace $X = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\}$. (Implying that for every φ, there exists a small neighbourhood of the point 0 on which $\hat{\varphi}$ vanishes.)

Hint 2: You can use without proof the **Method of Stationary Phase**: Let $\omega : \mathbb{R}^m \to \mathbb{R}$ in C^∞ and $u \in C_0^\infty(\mathbb{R}^n)$. Let G be an open neighbourhood of $\{\nabla\omega(k) : k \in \operatorname{supp}(u)\}$. Then for every $m \in \mathbb{N}$ there exists a c_m, such that

$$\left| \int_{\mathbb{R}^n} e^{i(k\cdot x - \omega(k)t)} u(k) \mathrm{d}k \right| \leq \frac{c_m}{(1+|t|)^m} \quad \text{for all } x,t \text{ with } x/t \notin G.$$

(Heuristically, the method of stationary phase is often described as saying that due to strong oscillations of the phase, different parts of the integral cancel out. The proof however just uses integration by parts. . .)

Problem 4: The Magnetic Schrödinger Operator (4+3+3 points)

Let $\mathbf{A} \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ be a divergence-free vector potential, i. e., $\operatorname{div} \mathbf{A} = \sum_{j=1}^3 \frac{\partial A_j}{\partial x_j} = 0$. We define the magnetic Schrödinger operator

$$H_{\mathbf{A}} := |-i\nabla + \mathbf{A}|^2 = -\Delta + \mathbf{A}^2 - \sum_{j=1}^3 2iA_j \partial_{x_j}.$$

a. Show that for every $\varepsilon > 0$ there exists $C(\varepsilon) < \infty$ such that $\|\mathbf{A}^2\varphi\| \leq \varepsilon\|-\Delta\varphi\| + C(\varepsilon)\|\varphi\|$. (We say that $\mathbf{A}^2$ is infinitesimally operator bounded w. r. t. the Laplacian.)

b. Show the same for $-2i\mathbf{A}\cdot\nabla = -\sum_{j=1}^3 2iA_j\partial_{x_j}$.

c. Use the Kato-Rellich theorem to conclude that $H_{\mathbf{A}}$ is self-adjoint on $H^2(\mathbb{R}^3)$.

Hint: You will need to use the following inequalities several times in combination and work out appropriate exponents (you don't need to prove these inequalities).

- Sobolev inequality: $\|f\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}$ for a constant C independent of $f \in H^1(\mathbb{R}^3)$.
- Hölder inequality: for all $p, q \in (1,\infty)$ with $p^{-1} + q^{-1} = 1$ and any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$
$$\int_{\mathbb{R}^d} |fg| \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$
- Young inequality: for all $p, q \in (1,\infty)$ with $p^{-1} + q^{-1} = 1$ and any $\varepsilon, a, b > 0$
$$ab \leq \frac{a^p}{\varepsilon^p p} + \frac{\varepsilon^q b^q}{q}.$$