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Advanced Mathematical Physics

Assignment 4 of 4

To be handed in at the beginning of the seminar on Friday, June 23. The assignment will be discussed on the same day, so we can not accept any submissions after the seminar has started!

Problem 1: Non-degenerate groundstate (10 points)

Let H be a semi-bounded self-adjoint operator in $L^2(\mathbb{R}^3)$, such that e^{-tH} is positivity improving. Let W be a self-adjoint Hilbert-Schmidt operator with negative integral kernel.

Show that, if $H + W$ has an eigenvalue at the bottom of its spectrum, this eigenvalue is non-degenerate and the corresponding eigenvector can be chose as a strictly positive function.

Problem 2: Hilbert-Schmidt operators (3+3+2+2 points)

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ and $K \in \mathcal{L}(\mathcal{H})$. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis[1] of $L^2(\mathbb{R}^d)$.

a. Show that $(e_{nm})_{n,m\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, where e_{nm} denotes

$$e_{nm}(x, y) := e_n(x)\overline{e_m(y)}, \quad x, y \in \mathbb{R}^d.$$

Hint: You can use the fact that the subspace of linear combinations of functions $f(x)g(y)$ with $f, g \in L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

b. Assume that K has an integral kernel $\tilde{K}(x, y)$ in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, i. e. for all $\psi \in L^2(\mathbb{R}^d)$

$$K\psi(x) = \int_{\mathbb{R}^d} \tilde{K}(x, y)\psi(y)\mathrm{d}y.$$

(Such operators K are called Hilbert-Schmidt operators.) Recall that the operator norm of K is smaller than the L^2-norm of its integral kernel.

Show that K is a compact operator.

Hint: You can use Parseval's relation in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and prove that K is the operator norm limit of a sequence of finite rank operators.

[1] A countable family $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called orthonormal basis if $\langle e_n, e_k\rangle = \delta_{n,k}$ and for all $x \in \mathcal{H}$ we have $x = \sum_{n=0}^{\infty} e_n\langle e_n, x\rangle$.

c. Let $f, g \in L^2(\mathbb{R}^d)$. Denote by $g(-i\nabla)$ operator $\mathscr{F}^{-1}T_g\mathscr{F}$ where T_g is the multiplication operator by g.
Show that the operator $f(x)g(-i\nabla)$ is Hilbert-Schmidt.

Problem 3: In confining potentials, there is only discrete spectrum. (3+2+2+2 points)

Let $V \geq 0$, $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and $\lim_{x\to+\infty} V(x) = +\infty$. Let $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$. You may take for granted that this is self-adjoint e. g. due to the Friedrich's extension.

For $c > 0$, let $R_c > 0$ be such that $V(x) \geq c$ for $|x| \geq R_c$. Let W be the potential $W := -c\mathbb{1}_{|x|<R_c}$ (where $\mathbb{1}_{|x|<R_c}$ is the characteristic function of the ball).

a. Prove that $\sigma_{\mathrm{ess}}(-\Delta + W) = \sigma_{\mathrm{ess}}(-\Delta)$.

b. Show that $V \geq c + W$ in the sense of operators.

Then deduce from this fact and **a.** that the min-max values μ_n satisfy

$$\mu_n(H) \geq c + \mu_n(-\Delta + W).$$

c. Prove that there exists an $n \in \mathbb{N}$ such that

$$\mu_n(H) \geq c - 1.$$

d. Conclude that $\lim_{n\to\infty} \mu_n(H) = \infty$ and that H has no essential spectrum.

Problem 4: A Lieb-Thirring inequality (2+3+3+2 points)

Let V be a real-valued potential that vanishes at infinity and satisfies $V \in L^{3/2}+L^\infty(\mathbb{R}^3)$. Furthermore, let $\gamma > 1/2$ and assume that the negative part of the potential satisfies $V_- \in L^{\gamma+3/2}(\mathbb{R}^3)$. Let $H := -\Delta + V$ and E_j its eigenvalues. Let $e > 0$.

a. Consider a Hilbert-Schmidt operator K with integral kernel $\tilde{K}(x, y)$ in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Let $\operatorname{tr} K^*K$ be the sum (including multiplicity) of all eigenvalues of the operator K^*K. Show that

$$\operatorname{tr} K^*K = \int |\tilde{K}(x, y)|^2 \mathrm{d}x\mathrm{d}y.$$

Hint: You can use without proof the singular-value decomposition: Any compact operator A on a Hilbert space can be written[2] as

$$A = \sum_{n=1}^{N} \sqrt{\lambda_n} |\varphi_n\rangle\langle\psi_n|,$$

[2]We use Dirac's notation for the operator $|\varphi_n\rangle\langle\psi_n| : f \mapsto \langle\psi_n|f\rangle\varphi_n$.

where $\{\psi_n\}_{n=1}^N$ is an orthonormal set of eigenvectors associated to the positive eigenvalues λ_n of A^*A. Furthermore $\{\varphi_n\}_{n=1}^N$ is the orthonormal set given by $\varphi_n = A\psi_n/\sqrt{\lambda_n}$. The sum can be finite or infinite, and converges in operator norm.

b. Let $K_e := V_-^{1/2}(-\Delta + e)^{-1}V_-^{1/2}$. Show that

$$\operatorname{tr} K_e^2 \leq \frac{C}{\sqrt{e}} \int V_-(x)^2 \mathrm{d}x.$$

c. Let N_e be the number (including multiplicity) of eigenvalues of H which are less than or equal to $-e$. Show that (with the sum running over all negative eigenvalues of H, including multiplicity)

$$\sum_{E_j<0} |E_j|^\gamma = \gamma \int_0^\infty e^{\gamma-1} N_e \mathrm{d}e.$$

Hint: $e^\gamma = \gamma \int_0^\infty x^{\gamma-1} \mathbb{1}_{[0,e]}(x)\mathrm{d}x$ (why?).

d. Show that there is a constant L_γ such that the sum over negative eigenvalues of H satisfies the Lieb-Thirring inequality

$$\sum_{E_j<0} |E_j|^\gamma \leq L_\gamma \int V_-(x)^{\gamma+3/2} \mathrm{d}x.$$

Hint: Check that and use $N_e(-V_-) = N_{e/2}(-V_- + \frac{e}{2}) \leq N_{e/2}(-(V + \frac{e}{2})_-)$. (The negative part of a function f is $f_-(x) := \max\{-f(x), 0\}$.)