SPECTRAL ANALYSIS OF (SOME) SCHRÖDINGER OPERATORS I

1. Introduction

We now turn to studying the spectrum of Schrödinger operators of the form $-\Delta + V$. For typical non-negative decaying potentials V (like the Coulomb potential $-\frac{1}{|\cdot|}$) the spectrum of the Hamiltonian $H := -\Delta + V$ can be decomposed as follows.

(1) The essential spectrum of H is that of $-\Delta$: $[0, +\infty)$.
(2) There is a finite or infinite number of negative eigenvalues with finite multiplicity below the essential spectrum: the corresponding eigenfunctions are called bound states.
(3) The bottom of the spectrum is a simple eigenvalue, the corresponding normalized eigenfunction (fixed up to a phase) is called the grond state.
(4) There is no positive eigenvalue.

The non-degeneracy of the ground state will be proved in a seminar, and the absence of positive eigenvalues will be established for certain kind of potentials later on[1].

Here, we introduce basic results for the study of a semi-bounded operator H: the min-max principle, and the Rayleigh-Ritz method. The first one characterizes the first elements of the spectrum of such an operator H. The second one is a technique to obtain *upper bounds* on these first eigenvalues. Both are natural, if not "obvious", yet very useful.

We will then give examples to illustrate the fact that the discrete spectrum can be finite or infinite. The crucial criterion is the behaviour of the potential V at infinity. Say $V \leq 0$ to simplify: if V decays slowly at infinity, then there exist smooth functions ψ with negative energy $\langle \psi, (-\Delta + V)\psi \rangle$ and arbitrarily far support.

The proof of the min-max principle is interesting as it plays with the definition of the discrete and infinite spectrum in terms of the projection-valued measure.

We emphasize that there exists another interesting regime: when the potential V tends to $+\infty$ as x tends to infinity. The best example is the (important) case of the harmonic oscillator with $V(x) = x^2$. In the fourth assignment you are asked to prove that in that case, there is no essential spectrum and that the discrete spectrum is made of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_n \to +\infty$.

In Physic textbooks, you can find a full description of the spectrum and the eigenfunctions of the atomic Schrödinger operator $-\frac{\Delta}{2} - \frac{Z}{|x|}$ and of the

[1] the key-notion to proving this result is: the virial theorem.

harmonic oscillator $-\Delta + |x|^2$. Both cases involve orthogonal polynomials: the Laguerre polynomials for the first one and the Hermite polynomials for the second.

2. The min-max principle

Theorem 1. *Let H be a semi-bounded s.a. operator. For $n \geq 1$ let $\mu_n(H)$ be:*

$$\begin{cases} \mu_n(H) := \sup\big\{U_H(\phi_1, \dots, \phi_{n-1}),\ \phi_1, \dots, \phi_{n-1} \in \mathcal{H}\big\}, \\ U_H(\phi_1, \dots, \phi_{n-1}) := \inf\big\{\langle \psi, H\psi\rangle,\ \psi \in \mathrm{dom}(H),\ \|\psi\|_{\mathcal{H}} = 1, \\ \qquad \&\ \forall\, 1 \leq i \leq n-1,\ \langle \phi_i, \psi\rangle = 0\big\}, \end{cases}$$

with the convention $\mu_1(H) := \inf\{\langle \psi, H\psi\rangle,\ \psi \in \mathrm{dom}(H),\ \|\psi\|_{\mathcal{H}} = 1\}$.

Then for each $n \geq 1$ there holds either one or the other of these two statements:

1. *There are n eigenvalues (counting multiplicity) below the bottom of the essential spectrum, and $\mu_n(H)$ is the n-th smallest eigenvalue (counting multiplicity).*
2. *There holds $\mu_n(H) = \inf \sigma_{\mathrm{ess}}(H)$, and for all $m \geq 1$ we have $\mu_{n+m}(H) = \mu_n(H)$. Furthermore there are at most $n-1$ eigenvalues (counting multiplicity) below $\mu_n(H)$.*

Remark 2. *Left as an exercise, one can show that* $\mathrm{dom}(H)$ *can be replaced by $\mathcal{Q}(H)$ up to replacing $\langle \psi, H\psi\rangle$ by $q_H(\psi, \psi)$.*

We emphasize that in the definition of the $\mu_n(H)$'s we do not put any condition on the family $(\phi_i)_{1 \leq i \leq n-1}$. In particular we may have $\phi_i = \phi_j$ and by construction we have the inequality $\mu_n(H) \leq \mu_{n+1}(H)$.

Try to apply it to the case where $\mathcal{H} = \mathbb{C}^N$ and H is an Hermitian matrix. In the definition of $\mu_n(H)$, what is the family $\phi_1, \dots, \phi_{n-1}$ which attains the supremum?

As a first application of the min-max principle we show the following.

Proposition 3. *Let $A \geq 0$ and B two s.a. operators. We assume that $\mathcal{Q}(A) \cap \mathcal{Q}(B)$ is dense and that, decomposing*

$$B = B_+ - B_- = B\chi_{[0,+\infty)}(B) + B\chi_{(-\infty,0)}(B),$$

the negative part B_- is infinitesimally form bounded w.r.t. A. Furthermore we assume that for all $\beta \geq 0$ we have:

$$\sigma_{\mathrm{ess}}(A + \beta B) = [0, +\infty).$$

Then for all $n \geq 1$, the function $\beta \in [0, +\infty) \mapsto \mu_n(A + \beta B)$ is monotone decreasing.

We emphasize that $A + \beta B$ denotes the s.a. operator obtained by the form sum and the KLMN theorem.

The idea is that we can compare the eigenvalues of two semi-bounded s.a. A, C operators through the expectations $q_A(\psi, \psi)$ and $q_C(\psi, \psi)$ provided we have some inclusion of the form domains. Of course in the proposition we have in mind the case $A = -\Delta$ and $B = V$.

Proof of the min-max principle. Let P_Ω be the projection valued measure of H.

We first show the following auxiliary result.

Auxiliary result. Let $a \in \mathbb{R}$.

- If $a < \mu_n$, then $\dim(\text{ran } P_{(-\infty,a)}) < n$.
- If $a > \mu_n$, then $\dim(\text{ran } P_{(-\infty,a)}) \geq n$.

We prove both claims by contraposition.

Assume $\dim(\text{ran } P_{(-\infty,a)}) \geq n$. Then in particular there exists an n-dimensional space $V \subset \text{ran } P_{(-\infty,a)}$. As H is bounded from below, then V is in its domain (the spectral measure associated to $\psi \in V$ has compact support).

So given any $\phi_1, \ldots, \phi_{n-1} \in \mathcal{H}$, the vector space $V \cap \{\phi_1, \ldots, \phi_{n-1}\}^\perp$ has dimension ≥ 1 and we can find ψ, $\|\psi\|_{\mathcal{H}} = 1$ in it. By construction (of V), there holds $\langle \psi, H\psi \rangle \geq a\|\psi\|_{\mathcal{H}}^2$. By definition of $\mu_n(H)$, we thus obtain: $\mu_n(H) \leq a$.

Similarly $\dim(\text{ran } P_{(-\infty,a)}) < n$. Then the range ran $P_{(-\infty,a)}$ is spanned by at most $n-1$ functions $\phi_1^{(0)}, \ldots, \phi_{n-1}^{(0)} \in \mathcal{H}$. Thus by construction we have:

$$\text{dom}(H) \cap \{\phi_1^{(0)}, \ldots, \phi_{n-1}^{(0)}\}^\perp \subset \text{ran } P_{[a,+\infty)}.$$

Take ψ in the above set: there holds $\langle \psi, H\psi \rangle \geq a\|\psi\|_{\mathcal{H}}^2$. Therefore $U_H(\phi_1^{(0)}, \ldots, \phi_{n-1}^{(0)}) \geq a$. By definition of $\mu_n(H)$ we thus obtain $\mu_n(H) \geq U_H(\phi_1^{(0)}, \ldots, \phi_{n-1}^{(0)}) \geq a$.

End of the proof. Observe that the fact that H is semi-bounde imply that the $\mu_n(H)$'s are all finite.

The fact that they are not equal to $+\infty$ is obvious (one can make the dichotomy $\mathcal{H}$ has finite dimension hence H is bounded or $\mathcal{H}$ has infinite dimensio but then for all $n \geq 1$ and all $\phi_1, \ldots, \phi_{n-1} \in \mathcal{H}$, the set $D(H) \cap \{\phi_1, \ldots, \phi_{n-1}\}^\perp$ has infinite dimension as well).

the fact that they are not equal to $-\infty$ follows from $\mu_1(H) \geq c$, where c is the bound of H: remember the inequalities $\mu_{n+1}(H) \geq \mu_n(H)$.

Having the definition of the discret and essential spectrum in mind, we make the dichotomy:

(1) either for all $\varepsilon > 0$, the range ran $P_{(-\infty,\mu_n+\varepsilon)}$ has infinite dimension,
(2) or there is some $\varepsilon_0 > 0$ for which ran $P_{(-\infty,\mu_n+\varepsilon_0)}$ has finite dimension.

We claim that if the first case holds, then we are in the second case of Thm 1, and if the second case holds, then we are in the first case of Thm 1.

Assume that for all $\varepsilon > 0$, the range ran $P_{(-\infty,\mu_n+\varepsilon)}$ has infinite dimension. Take $\varepsilon > 0$, we know that $\dim \text{ran } P_{(-\infty,\mu_n-\varepsilon]} \leq \dim \text{ran } P_{(-\infty,\mu_n-\varepsilon/2)} \leq n-1$. Hence we have:

$$\dim \text{ran } P_{(\mu_n-\varepsilon,\mu_n+\varepsilon)} = +\infty,$$

and as $\varepsilon > 0$ was arbitrary we obtain $\mu_n \in \sigma_{\text{ess}}(H)$. If $a < \mu_n$, then there exists $\varepsilon > 0$ with $a < a + \varepsilon < \mu_n$, and:

$$\dim \text{ran } P_{(a-\varepsilon,a+\varepsilon)} \leq \dim \text{ran } P_{(-\infty,a+\varepsilon)} \leq n-1.$$

So by definition $a \notin \sigma_{\text{ess}}(H)$, and μ_n is the bottom of the essential spectrum.

Necessarily $\mu_{n+1} = \mu_n$, else we would have:

$$\dim \operatorname{ran} P_{(-\infty, 2^{-1}(\mu_n+\mu_{n+1}))} \leq n+1.$$

Assume that there is some $\varepsilon_0 > 0$ for which ran $P_{(-\infty,\mu_n+\varepsilon_0)}$ has finite dimension. By the auxiliary result, for all $\varepsilon > 0$ we have:

$$\dim \operatorname{ran} P_{(\mu_n-\varepsilon,\mu_n+\varepsilon)} \geq \dim \operatorname{ran} P_{(-\infty,\mu_n+\varepsilon)} - \dim \operatorname{ran} P_{(-\infty,\mu_n-\varepsilon]} \geq n-(n-1) = 1.$$

Hence $\mu_n \in \sigma(H)$. As

$$\dim \operatorname{ran} P_{(\mu_n-\varepsilon_0,\mu_n+\varepsilon_0)} \leq \dim \operatorname{ran} P_{(-\infty,\mu_n+\varepsilon_0)} < +\infty,$$

we have $\mu_n \in \sigma_{\mathrm{disc}}(H)$. In particular there exists $\delta > 0$ such that

$$(\mu_n - \delta, \mu_n + \delta) \cap \sigma(H) = \{\mu_n\}.$$

So we have: dim ran $P_{(-\infty,\mu_n]} =$ dim ran $P_{(-\infty,\mu_n+\delta)} \geq n$, and there exists at least n eigenvalues $E_1 \leq E_2 \leq \cdots \leq E_n \leq \mu_n$. Necessarily $E_n = \mu_n$, else we would have: dim ran $P_{(-\infty,E_n]} = n$ contradicting the auxiliary result.

□

Proof of Proposition 3. Let $n \geq 1$ and $\beta \geq 0$.

By assumption the essential spectrum is always $[0, +\infty)$, hence $\mu_n(A + \beta B) \leq 0$ coincides with:

$$\mu_n(A+\beta B) = \sup_{\phi_1,\dots,\phi_{n-1}\in\mathcal{H}} \min\left\{ \min(0, q_{A+\beta B}(\psi,\psi)),\ \psi \in \mathcal{Q}(A)\cap\mathcal{Q}(B),\ \|\psi\|_{\mathcal{H}} = 1\right\}.$$

Let $0 \leq \beta_1 \leq \beta_2$. For every $\psi \in \mathcal{Q}(A) \cap \mathcal{Q}(B)$, $\|\psi\|_{\mathcal{H}} = 1$:

(1) either $q_B(\psi,\psi) \geq 0$ but then $q_{A+\beta B}()\psi,\psi) \geq 0$ for all $\beta \geq 0$,
(2) or $q_B(\psi,\psi) < 0$ but then:

$$q_{A+\beta_2 B}(\psi,\psi) < q_{A+\beta_1 B}(\psi,\psi).$$

Considering the formula of μ_n, we thus obtain $\mu_n(A+\beta_2 B) \leq \mu_n(A+\beta_1 B)$.

□

3. The Rayleig-Ritz method

3.1. **The method.** This method corresponds to approximating the first eigenvalues of a semi-bounded operator by restricting the min-max scheme to a finite dimensional subspace of the domain.

It is immediate that we obtain so an upper bound of the ground state.

Theorem 4. *Let H be a semi-bounded s.a. operator and let $V \subset \operatorname{dom}(H)$ be an N-dimensional subspace, $N \in \mathbb{N}$. Let P_V be the orthogonal projection onto V and $H_V := P_V H P_V$.*

Let $\hat{\mu}_1 \leq \hat{\mu}_2 \leq \dots \hat{\mu}_N$ be the eigenvalues of the restriction $(H_V)_{|V} : V \to V$. Then there holds:

$$\forall\, 1 \leq n \leq N,\ \mu_n(H) \leq \hat{\mu}_n.$$

In particular we have the following.

Corollary 5. *Under the assumption of the previous theorem: if H has M eigenvalues below the essential spectrum $E_1 \leq E_2 \leq \ldots \leq E_M$ then for all $1 \leq k \leq \min(N, M)$ we have $E_k \leq \hat{\mu}_k$.*

Furthermore, let $(\phi_n)_{n\in\mathbb{N}}$ is a Hilbert basis made of elements of $\mathrm{dom}(H)$, and assume that $\mu_1(H)$ is an eigenvalue with normalized eigenfunction $\psi = \sum_n a_n\phi_n$. Let P_N be the orthogonal projection onto the span of the ϕ_n, $1 \leq n \leq N$.

If $\lim_{N\to+\infty}\langle P_N\psi, HP_N\psi\rangle = \mu_1(H)$ then the lowest eigenvalue $\hat{\mu}_1^{(N)}$ of the matrix $(\langle\phi_n, H\phi_m\rangle)_{1\leq n,m\leq N}$ converges to $\mu_1(H)$ as $N \to +\infty$.

For the latter result, the matrix corresponds to the restriction of $P_N HP_N$ to ran P_N, written in the basis $(\phi_1, \ldots, \phi_N)$. If $\lim_{N\to+\infty}\langle P_N\psi, HP_N\psi\rangle = \mu_1(H)$, then as $N \to +\infty$ we get by the Rayleigh-Ritz method:

$$\mu_1(H) \leq \hat{\mu}_1^{(N)} \leq \frac{\langle P_N\psi, HP_N\psi\rangle}{\langle\psi, P_N\psi\rangle} \underset{n\to+\infty}{\longrightarrow} \mu_1(H).$$

Proof of Thm 4. By the min-max principle applied to the restriction of H_V to V, we have:

$$\begin{aligned}
\hat{\mu}_m := & \sup_{\substack{\phi_i\in V\\ 1\leq i\leq m-1}} \inf_{\substack{\psi\in V,\, \|\psi\|_{\mathcal{H}}=1\\ \langle\phi_i,\psi\rangle=0}} \langle\psi, H\psi\rangle,\\
= & \sup_{\substack{\phi_i\in \mathcal{H}\\ 1\leq i\leq m-1}} \inf_{\substack{\psi\in V,\, \|\psi\|_{\mathcal{H}}=1\\ \langle\mathbf{P_V}\phi_i,\psi\rangle=0}} \langle\psi, H\psi\rangle,\\
= & \sup_{\substack{\phi_i\in V\\ 1\leq i\leq m-1}} \inf_{\substack{\psi\in V,\, \|\psi\|_{\mathcal{H}}=1\\ \langle\phi_i,\psi\rangle=0}} \langle\psi, H\psi\rangle,\\
\geq & \sup_{\substack{\phi_i\in V\\ 1\leq i\leq m-1}} \inf_{\substack{\psi\in \mathcal{H},\, \|\psi\|_{\mathcal{H}}=1\\ \langle\phi_i,\psi\rangle=0}} \langle\psi, H\psi\rangle.
\end{aligned}$$

For the third line, we have used the fact that for $\psi \in V$ and $\phi_i \in \mathcal{H}$ there holds $\langle P_V\phi_i, \psi\rangle = \langle\phi_i, \psi\rangle$. □

3.2. **First example.** Remember the Hardy inequality in $\mathbb{R}^3$: for all $\psi \in H^1(\mathbb{R}^3)$, we have:

$$\int \frac{|\psi(x)|^2}{|x|^2}\mathrm{d}x \leq 4\int |\nabla\psi(x)|^2\mathrm{d}x.$$

It implies, as we have seen, that for all $0 < \alpha < 2$, the quadratic form defined by $|x|^{-\alpha}$ is infinitesimally form bounded w.r.t. $-\Delta$. Hence using the KLMN theorem, we get that for all $a \in \mathbb{R}$, $0 < b < 4^{-1}$, and $0 < \alpha < 2$ the following operators are self-adjoint with form domain $H^1(\mathbb{R}^3)$:

$$-\Delta - \frac{a}{|x|^\alpha} \quad \& \quad -\Delta - \frac{b}{|x|^2}.$$

For the latter operator, Hardy's inequality even ensures us that it has no negative eigenvalue.

As for the other, we have the following.

Proposition 6. *Let $V = V_R + V_\infty$ with $(V_R, V_\infty) \in \mathcal{R} \times L^\infty(\mathbb{R}^3)$ and $\lim_{x\to+\infty} V_\infty(x) = 0$.*

(1) *If there exist* $R_0, a, \varepsilon > 0$ *such that for all* $x \in \mathbb{R}^3$, $|x| \geq R_0$ *there holds:*
$$V(x) \leq -a|x|^{-2+\varepsilon},$$
then $\sigma_{\rm disc}(-\Delta + V)$ *is infinite.*

(2) *If there exist* $R_0 > 0$ *and* $b < 1$ *such that for all* $x \in \mathbb{R}^3$, $|x| \geq R_0$ *there holds:*
$$V(x) \geq -\frac{b}{4}|x|^{-2},$$
then $\sigma_{\rm disc}(-\Delta + V)$ *is finite.*

The condition on the V_∞ is to ensure that the essential spectrum of $-\Delta+V$ is $[0, +\infty)$.

Lemma 7. *Let* $f, g \in L^\infty(\mathbb{R}^d)$ *such that* $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = 0$. *Writing down* T_f *the multiplication operator by* f *and* $\mathscr{F}$ *the Fourier transform, then the operator:*
$$f(x)g(-i\nabla) := T_f\mathscr{F}^{-1}T_g\mathscr{F}$$
is compact.

The Lemma is proven at the very end.

Here for $C > 0$ large enough, we have:
$$(-\Delta + V + C)^{-1} - (-\Delta + C)^{-1} = (-\Delta + C)^{-1}(V_R + V_\infty)(-\Delta + V + C)^{-1}. \tag{1}$$

We have shown previously that $(-\Delta + C)^{-1}V_R(-\Delta + V + C)^{-1}$ is compact. Here $(-\Delta + C)^{-1}V_\infty = g(-i\nabla)V_\infty(x)$ is compact as $V_\infty(x), g(p) = (p^2 + C)^{-1} \in L^\infty(\mathbb{R}^3)$ both tend to zero at infinity.

Remark 8. *For the second part, we will use a result which will be proven later on: the fact that if* $V \in \mathcal{R}$ *then* $-\Delta + V$ *has only finitely many bound states.*

Proof of Proposition 6. Let us prove the first part.

We have seen that $\sigma_{\rm ess}(-\Delta + V) = \sigma_{\rm ess}(-\Delta) = [0, +\infty)$. So it suffices to show that for all $n \in \mathbb{N}$ there holds $\mu_n(-\Delta + V) < 0$. To do so we will apply the Rayleigh-Ritz method to well-chosen subspaces.

The key-idea is that under the dilation: $\psi(x) \mapsto \lambda^{-3/2}\psi(\lambda^{-1}x)$, which is unitary transformation,

(1) the kinetic energy $\|\nabla\psi\|_{L^2}^2$ scales like λ^{-2},
(2) while the potential energy $-\langle\psi, |\cdot|^{-2+\varepsilon}\psi\rangle$ scales like $\lambda^{-2+\varepsilon}$.

As $\varepsilon > 0$, for a given ψ, the second term is bigger (in absolute value) as $\lambda \to +\infty$. So we just have to construct a countable family $(\psi_n)_n$ with disjoint supports for which this behaviour hold.

Let $\psi \in C_0^\infty(\mathbb{R}^3)$ different from 0 and such that
$$\|\psi\|_{L^2} = 1 \quad \& \quad \text{supp } \psi \subset \big\{x \in \mathbb{R}^3,\ 1 < |x| < 2\big\}.$$
In particular for $\lambda > 0$ we have:
$$\|\psi_\lambda\|_{L^2} = 1 \quad \& \quad \text{supp } \psi_\lambda \subset \big\{x \in \mathbb{R}^3,\ \lambda < |x| < 2\lambda\big\}.$$

We choose $R_1 > R_0$ big enough such that ψ_{R_1} satisfies:

$$\begin{aligned}\langle \psi_{R_1}, (-\Delta + V)\psi_{R_1}\rangle &= \int |\nabla \psi_{R_1}|^2 + \int V|\psi_{R_1}|^2,\\ &\le \int |\nabla \psi_{R_1}|^2 - a\int \frac{|\psi_{R_1}|^2}{|\cdot|^{2-\varepsilon}},\\ &\le R_1^{-2}\int |\nabla\psi|^2 - aR_1^{-2+\varepsilon}\frac{|\psi|^2}{|\cdot|^{2-\varepsilon}} < 0.\end{aligned}$$

We then consider the family $(\psi_{2^n R_1})_{n\ge 1}$: it is an *orthonormal* family, and as $-\Delta + V$ is a local operator we also have $\langle \psi_n, (-\Delta+V)\psi_m\rangle$ for $n \neq m$ (as the two functions have disjoint supports). As for ψ_{R_1}, we have:

$$\langle \psi_{2^n R_1}, (-\Delta + V)\psi_{2^n R_1}\rangle \le (2^n R_1)^{-2}\int |\nabla\psi|^2 - a(2^n R_1)^{-2+\varepsilon}\frac{|\psi|^2}{|\cdot|^{2-\varepsilon}} < 0.$$

Applying the Rayleigh-Ritz method to $V_N := \operatorname*{span}_{1\le n\le N}(\psi_{2^n R_1})$ for all $N \ge 1$, we obtain that for all $n \in \mathbb{N}$ there holds:

$$\mu_n(-\Delta + V) < 0.$$

We now turn to the second part. Let $W_0(x) := V(x) + \frac{b}{4}\min(1, |x|^{-2})$. By assumption, for $|x| \ge \max(R, 1)$, there holds $W_0(x) \ge 0$, hence the function $W := \min(W_0, 0)$ has compact support.

Furthermore, as $V \in \mathcal{R} + L^\infty$ and $\frac{b}{4}\min(1, |x|^{-2}) \in L^\infty(\mathbb{R}^3)$, then W_0 and W are both in $\mathcal{R} + L^\infty$. Since W has compact support, then there holds $W \in \mathcal{R}$: it suffices to write $W = W_R + W_\infty$ and to observe that $W_R \mathbb{1}_{|x|\le R+1} \in \mathcal{R}$ and $W_\infty \mathbb{1}_{|x|\le R+1} \in \mathcal{R}$.

We have the following inequality of quadratic forms on $\mathcal{Q}(-\Delta) = H^1(\mathbb{R}^3)$:

$$\begin{aligned}-\Delta + V &= -(1-b)\Delta + W_0 + b\Big(-\Delta - \frac{1}{4|\cdot|^2}\Big),\\ &\ge -(1-b)\Delta + W_0,\\ &\ge -(1-b)\Delta + W.\end{aligned}$$

By the min-max principle, we obtain:

$$\mu_n(-\Delta + V) \ge \mu_n(-(1-b)\Delta + W) = (1-b)\mu_n(-\Delta + (1-b)^{-1}W).$$

As $W \in \mathcal{R}$, then $-\Delta + (1-b)^{-1}W$ has only finitely many bound states and $\sigma_{\text{ess}}(-\Delta + (1-b)^{-1}W) = [0, +\infty)$. Let $N_0 \ge 0$ be the number of bound states: for $n \ge N_0 + 1$ there holds:

$$0 \ge \mu_n(-\Delta + V) \ge (1-b)\mu_n(-\Delta + (1-b)^{-1}W) = 0.$$

□

Proof of Lemma 7. You are asked in the fourth assigment to show that if $F, G \in L^2(\mathbb{R}^d)$, then the operator $F(x)G(-i\nabla)$ is compact (and even Hilbert-Schmidt).

Here consider $R > 0$ and let $f_R(x) := f(x)\mathbb{1}_{|x|\le R}$ and $g_R(x) := g(x)\mathbb{1}_{|x|\le R}$. For any $R > 0$, we have $f_R, g_R \in L^2(\mathbb{R}^d)$, hence $f_R(x)g_R(-i\nabla)$ is compact. As $\lim_{R\to+\infty}\|f(x) - f_R(x)\|_{\mathcal{L}} = \lim_{R\to+\infty}\|g(-i\nabla) - g_R(-i\nabla)\|_{\mathcal{L}} = 0$ by the assumption on the decay of the functions, we get that $f(x)g(-i\nabla)$ is compact as an operator-norm limit of compact operators. $\square$

References

[1] Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, New York-London, 1972.

[2] Gerald Teschl, *Mathematical methods in quantum mechanics*, 2nd ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. With applications to Schrödinger operators.