THE SPECTRAL THEOREM

1. Introduction

In this part, we will give the different forms of the spectral theorem for self-adjoint A unbounded operators on a Hilbert space $\mathcal{H}$. This theorem is the generalization of the well-known result in finite dimension which states that every Hermitian matrix (or more generally every normal matrix) can be diagonalized in an orthonormal basis.

In the unbounded setting, it is convenient to use the functional calculus form, which enables us to take the function $f(A)$ of a s.a. operator A with f a Borelian and bounded function.

In the case of a bounded operator (not necessarily s.a.) we can also consider the *analytical* functional calculus, using the Cauchy formula to define a new operator. We refer the reader to the given references.

Throughout this part, A denotes a s.a. operator in a Hilbert space $\mathcal{H}$. For simplicity (essentially in Section 3), we will assume that it is separable, that is that it admits a dense sequence, or equivalently that it admits a countable Hilbert basis.

We take as a starting point the S.C.U.G. $(e^{itA})_{t\in\mathbb{R}}$ characterizing A and show that it gives rise to the functional calculus [1].

We will use two big theorems of measure theory: the theorem of Radon-Nikodym-Lebesgue and the theorem of Riesz-Markov.

It is not important to know them or their proofs, but one must realize that a big machinery is used for the proof (Riesz Markov in Section 2) and the decomposition of the spectrum into absolutely continuous, continuous singular and pure point spectrum (Lebesgue-Radon-Nikodym in Section 3).

Remark 1 (To be remembered)**.** *The important parts of this lecture are*[2]

1. *the functional calculus form of the spectral theorem (thm 2),*
2. *the spectral mapping of the spectrum (thm 6),*
3. *the decomposition of the spectrum into discrete and essential spectrum (section 4.2),*
4. *and at last, the Weyl's criterion for the essential spectrum and the theorem of stability for the essential spectrum thms 14 and 16.*

2. Functional calculus

2.1. **Statement.** We start with some notations.

[1] We have shown previously in the lecture the one-to-one correspondence between s.a. operators and S.C.U.G. in $\mathcal{H}$ independently from the spectral theorem. In [1], the authors first prove the spectral theorem before talking about S.C.U.G.

[2] if not everything

Notation 1. *The set of Borel sets on* $\mathbb{R}$ *will be denoted by* $\mathrm{Bor}(\mathbb{R})$*, and the set of* bounded *Borel functions by* $\mathcal{B}(\mathbb{R})$*.*

We say that a sequence $(f_k)_k$ *in* $\mathcal{B}(\mathbb{R})$ *converges to* f *if*

(1) $\sup_k \sup_{x\in\mathbb{R}} |f_k(x)| < +\infty$ *and*
(2) $f_k(x) \to f(x)$ *for all* $x \in \mathbb{R}$*, and* ***not*** *just almost everywhere.*

We aim to prove the following.

Theorem 2. *[Functional calculus] Let* A *s.a. on* $\mathcal{H}$*. Then there exists a unique map* $\widehat{\phi} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ *such that the following five points hold.*

(1) $\widehat{\phi}$ *is an algebraic* $*$*-homomorphism.*
(2) $\widehat{\phi}$ *is norm-continuous:* $\|\widehat{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} \le \sup_{x\in\mathbb{R}} |f(x)|$*.*
(3) *If a sequence* $(f_k)_k$ *of* $\mathcal{B}(\mathbb{R})$ *satisfies* $|f_k(x)| \le |x|$ *and* $\lim_{k\to+\infty} f_k(x) = x$ *for all* x*, then for all* $\psi \in \mathrm{dom}(A)$ *there holds:*
$$\lim_{k\to+\infty} \|\widehat{\phi}(f_k)\psi - A\psi\|_{\mathcal{H}} = 0.$$
(4) *If* $(f_k)_k$ *converges to* f *in* $\mathcal{B}(\mathbb{R})$ *then*
$$\operatorname*{s.lim}_{k\to+\infty} \widehat{\phi}(f_k) = \widehat{\phi}(f).$$
(5) *If* $f \ge 0$ *then* $\widehat{\phi}(f) \ge 0$ *as an operator and if* $A\psi = \lambda\psi$ *then* $\widehat{\phi}(f)\psi = f(\lambda)\psi$*.*

Remark 3. *Later on, the functional calculus will be written* $f(A)$ *instead of* $\widehat{\phi}(f)$*.*

Remark 4. *In the case where* $A = -\Delta$ *the operator* $f(-\Delta)$ *is a special case of Fourier multipliers. If we denote by* $\mathscr{F}$ *the Fourier transform, there holds:*
$$f(-\Delta) := \mathscr{F}^{-1} f(|p|^2)\mathscr{F} : \psi(x) \mapsto \mathscr{F}^{-1}\big(p \mapsto f(|p|^2)\mathscr{F}(\psi)(p)\big).$$
For the Fourier transform, we took the convention used previously.

2.2. **Proof.** We prove the theorem stepwise. We first define the functional calculus on functions in the Schwartz class $\mathcal{S}(\mathbb{R})$ and show the norm-conintuity in that case. By density this allows us to extend the functional calculus to the set $C_0(\mathbb{R})$ of continuous functions which converges to 0 at infinity[3] We extend at last the functional calculus to $\mathcal{B}(\mathbb{R})$ using the Riesz-markov theorem.

2.2.1. *Determination of* $\widehat{\phi}(e^{itx})$. Let us say that such a map exists. For $t \in \mathbb{R} \setminus \{0\}$, we consider the function
$$h_t(x) := \frac{e^{itx} - 1}{t}. \tag{1}$$
By the mean value-thorem we have:
$$|h_t(x)| \le |x| \sup_{y\in[0,x]} \big|\tfrac{\mathrm{d}}{\mathrm{d}s}(e^{is})_{|s=y}\big| \le |x|,$$
and $\lim_{\substack{t\to0\\ t\neq0}} h_t(x) = ix$. By point 3. of Thm 2 and Stone theorem we get that necessarily $\widehat{\phi}(e^{itx}) = e^{itA}$.

[3]As we have $C_0(\mathbb{R})$ is the closure of $\mathcal{S}(\mathbb{R})$ in the $L^\infty(\mathbb{R})$-norm.

2.2.2. *Case $f \in \mathcal{S}(\mathbb{R})$.* Consider the (other convention of the) Fourier transform for $f \in \mathcal{S}(\mathbb{R})$:

$$\hat{f}(t) := \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-itx}\mathrm{d}x,$$

with inverse Fourier transform:

$$f(x) = \int_{\mathbb{R}} \hat{f}(t)e^{itx}\mathrm{d}t. \tag{2}$$

If such a map $\widehat{\phi}$ exists, then applying it to (2) yields:

$$\widehat{\phi}(f) = \int_{\mathbb{R}} \hat{f}(t)\phi(e^{itx})\mathrm{d}t = \int_{\mathbb{R}} \hat{f}(t)e^{iAt}\mathrm{d}t. \tag{3}$$

Conversely we check that (3) defines an algebraic $*$-homomorphism. First, as the e^{itA} are unitary, dominated convergence[4] ensures that (3) is well-defined as we have:

$$\int \|\hat{f}(t)e^{itA}\|_{\mathcal{L}(\mathcal{H})}\mathrm{d}t \leq \int |\hat{f}(t)|\mathrm{d}t < +\infty.$$

Then it is clear that it defines a linear map: for $f, g \in \mathcal{S}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ we have:

$$\widehat{\phi}(\lambda f + g) = \lambda\widehat{\phi}(f) + \widehat{\phi}(g).$$

A computation (involving Fubini theorem) yieds:

$$\begin{aligned}
\widehat{\phi}(f)\widehat{\phi}(g) &= \iint \hat{f}(t)\hat{g}(s)e^{iA(t+s)}\mathrm{d}t\mathrm{d}s, \\
&= \int_t e^{iAt}\left[\int_s \hat{f}(t-s)\hat{g}(s)\mathrm{d}s\right]\mathrm{d}t, \\
&= \int_t e^{iAt}\widehat{[fg]}(t)\mathrm{d}t, \\
&= \widehat{\phi}(fg).
\end{aligned}$$

Similarly we have:

$$\begin{aligned}
\widehat{\phi}(f)^* &= \int \left(e^{itA}\hat{f}(t)\right)^*\mathrm{d}t, \\
&= \int e^{-itA}\overline{\hat{f}}(t)\mathrm{d}t, \\
&= \int e^{-itA}\hat{\overline{f}}(-t)\mathrm{d}t, \\
&= \int e^{itA}\hat{\overline{f}}(t)\mathrm{d}t = \widehat{\phi}(\overline{f}).
\end{aligned}$$

This shows that (3) defines an algebraic $*$-homomorphism on $\mathcal{S}(\mathbb{R})$.

In particular for f Schwartz, we have $|f|^2 = \overline{f}f$, hence:

$$\widehat{\phi}(|f|^2) = \widehat{\phi}(\overline{f}f) = \widehat{\phi}(\overline{f})\widehat{\phi}(f) = \widehat{\phi}(f)^*\widehat{\phi}(f) \geq 0.$$

[4]For operator-valued functions!

2.2.3. *Extension to* $C_0(\mathbb{R})$. We now prove that $\widehat{\phi}$ is norm-continuous on $\mathcal{S}(\mathbb{R})$. By density this will show that we can extend this morphism to $C_0(\mathbb{R})$. Let $f \in \mathcal{S}(\mathbb{R})$. Up to considering $f/((1+\varepsilon)\|f\|_{L^\infty})$ at taking the limit $\varepsilon \to 0^+$, it suffices to prove:

$$\|f\|_{L^\infty} < 1 \;\Rightarrow\; \|\widehat{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} \leq 1.$$

Consider such an f, then $g := f\sqrt{1-|f|^2}$ is also Schwartz, and we have:

$$0 \leq \widehat{\phi}(|g|^2) = \widehat{\phi}(|f|^2 - |f|^4),$$

hence

$$0 \leq \widehat{\phi}(|f|^2)^2 \leq \widehat{\phi}(f)^*\widehat{\phi}(f).$$

taking the expectation against ψ gives:

$$\begin{aligned}\|\widehat{\phi}(|f|^2)\psi\|_{\mathcal{L}}^2 &= \sup_{\|\psi\|_{\mathcal{H}}=1} \langle \psi, \widehat{\phi}(|f|^2)^2\psi\rangle,\\ &\leq \sup_{\|\psi\|_{\mathcal{H}}=1} \langle \psi, \widehat{\phi}(f)^*\widehat{\phi}(f)\psi\rangle \leq \|\widehat{\phi}(f)\|_{\mathcal{L}}^2.\end{aligned}$$

As $\|\widehat{\phi}(|f|^2)^2\|_{\mathcal{L}} = \|\widehat{\phi}(|f|^2)\|_{\mathcal{L}}^2 = \|\widehat{\phi}(f)\|_{\mathcal{L}}^4$ we get: $\|\widehat{\phi}(f)\|_{\mathcal{L}} \leq 1$.

We have used the fact that if B is bounded s.a. then:

$$\|B\|_{\mathcal{L}} = \sup_{\|\psi\|_{\mathcal{H}}=1} |\langle \psi, B\psi\rangle| \quad \& \quad \|B^*B\|_{\mathcal{L}} = \|B\|_{\mathcal{L}}^2.$$

(**Try to prove both statements, for the first one you may use Weyl sequences**). We have proved:

$$\forall\, f \in \mathcal{S}(\mathbb{R}),\; \|\widehat{\phi}(f)\|_{\mathcal{L}} \leq \|f\|_{L^\infty}. \tag{4}$$

By density this shows that $\widehat{\phi}$ can be *uniquely* extended [5] to $C_0(\mathbb{R})$ to a norm-continuous morphism satisfying (4).

2.2.4. *Extension to* $\mathcal{B}(\mathbb{R})$. In this part, we use Riesz-Markov theorem. Given $\psi \in \mathcal{H}$, for all $f \in C_0(\mathbb{R})$ there holds $|\langle \psi, \widehat{\phi}(f)\psi\rangle| \leq \|f\|_{L^\infty}\|\psi\|_{\mathcal{H}}^2$ and, if $f \geq 0$, there holds

$$\langle \psi, \widehat{\phi}(f)\psi\rangle = \langle \psi, \widehat{\phi}(f^{1/2})^*\widehat{\phi}(f)^{1/2}\psi\rangle \geq 0.$$

Therefore the map

$$f \in C_0(\mathbb{R}) \mapsto I_\psi(f) := \langle \psi, \widehat{\phi}(f)\psi\rangle$$

is a *positive* (hence continuous[6]) linear form on $C_0(\mathbb{R})$ with:

$$\sup_{0\leq f\leq 1} I_\psi(f) \leq \|\psi\|_{\mathcal{H}}^2 \times 1 < +\infty.$$

By Riesz-Markov theorem, this implies that there exists a *finite (regular) Borel* measure $\mu_\psi : \mathrm{Bor}(\mathbb{R}) \to [0,+\infty)$ such that:

$$I_\psi(f) = \int f(x)\mathrm{d}\mu_\psi(x).$$

[5] if $(f_n)_n$ in Schwartz converges to $f \in C_0(\mathbb{R})$, then it is a Cauchy sequence in L^∞ and $(\widehat{\phi}(f_n))$ is a Cauchy sequence in $\mathcal{L}$. The limit of $\widehat{\phi}(f_n)$ then exists and is independent from the sequence $f_n \to f$.

[6] here, we can separately obtain continuity.

The measure $d\mu_\psi$ is called the spectral measure for A (associated to ψ) and will be more thoroughly studied later. We emphasize that μ_ψ has finite mass:

$$\mu_\psi(\mathbb{R}) = \sup_{\substack{f\in C_0(\mathbb{R})\\ 0\le f\le 1}} \int f d\mu_\psi \le \|\psi\|^2_{\mathcal{H}}.$$

Remark 5. *We will not go into details about this theorem which is quite subtle. It is a representation lemma, like the Riesz Lemma, and identify the positive linear forms of $C_c(X)$ (continuous with compact support), when X is a locally compact Hausdorff space.*

A consequence is a representation theorem for the dual of $C_0(X)$. Here X is simply $\mathbb{R}$. We just emphasize that given a positive linear form ℓ on $C_0(\mathbb{R})$, the associated measure on a bounded open set U is given by

$$\ell(U) = \sup\big\{\ell(U),\ f\in C_c(\mathbb{R}),\ 0\le f\le 1,\ \mathrm{supp}(f)\subset U\big\}.$$

Observing that a bounded operator B is uniquely determined by its entries $\langle\phi, B\psi\rangle$, this representation enables us to extend $\widehat{\phi}$ to Borelian functions by *polarizing* $I_\psi(f)$ (in ψ). For $\psi,\phi\in\mathcal{H}$ and $f\in C_0(\mathbb{R})$ we have indeed:

$$\begin{aligned}\langle\psi,\widehat{\phi}(f)\psi\rangle =: I_{\phi,\psi}(f) &= \tfrac14\Big[\langle(\phi+\psi),\widehat{\phi}(f)(\phi+\psi)\rangle - \langle(\phi-\psi),\widehat{\phi}(f)(\phi-\psi)\rangle \\ &\quad +\tfrac1i(\langle(\phi+i\psi),\widehat{\phi}(f)(\phi+i\psi)\rangle - \langle(\phi-i\psi),\widehat{\phi}(f)(\phi-i\psi)\rangle)\Big],\\ &= \tfrac14\big(I_{\phi+\psi}(f) - I_{\phi-\psi}(f) + \tfrac1i(I_{\phi+i\psi}(f) - I_{\phi-i\psi}(f))\big).\end{aligned}$$

In other words we have:

$$\langle\psi,\widehat{\phi}(f)\psi\rangle = \frac14\Big[\int f\big(d\mu_{\phi+\psi} - d\mu_{\phi-\psi} + \tfrac1i(d\mu_{\phi+i\psi}) - d\mu_{\phi-i\psi})\big)\Big] \tag{5}$$

The R.H.S. of 5 is well-defined for $\mathcal{B}(\mathbb{R})$, and defines $I_{\phi,\psi}(f)$. If we pick a sequence $(f_k)_k$ in $C_0(\mathbb{R})$ which converges to $f\in\mathcal{B}(\mathbb{R})$ in the sense of Notation 1, and using the norm-estimate on $C_0(\mathbb{R})$ on the L.H.S. we get that

$$(\phi,\psi)\in\mathcal{H}^2\mapsto I_{\phi,\psi}(f),$$

is a *bounded* sesquilinear form on $\mathcal{H}$ with norm smaller than $\limsup_{k\to+\infty}\|f_k\|_{L^\infty}$, which is finite. By using a mollifier, we can show that the norm is in fact smaller than $\|f\|_{L^\infty}$. By Riesz representation Lemma, there exists a *unique* bounded operator $\widehat{\phi}(f)$ such that we have

$$I_{\phi,\psi}(f) = \langle\phi,\widehat{\phi}(f)\psi\rangle.$$

We thus have extended the functional calculus to $\mathcal{B}(\mathbb{R})$. That it is an algebraic $*$-homomorphism is easy to show with sequences in $C_0(\mathbb{R})$ and is left to the reader. The dominated convergence implies that this extension is unique if we request weak operator continuity[7].

[7]that is if we request that the maps $f\in\mathcal{B}(\mathbb{R})\mapsto\langle\phi,\widehat{\phi}(f)\psi\rangle$ are continuous for all $\phi,\psi\in\mathcal{H}$.

2.2.5. *End of the proof.* We first study the strong continuity of the functional calculus (fourth point). Let $f_k \to f$ in $\mathcal{B}(\mathbb{R})$. By dominated convergence in (5), we get that for all $\phi, \psi \in \mathcal{H}$ we have:

$$\langle \phi, \widehat{\phi}(f_n)\psi\rangle \to \langle \phi, \widehat{\phi}(f)\psi\rangle,$$

in other words $(\widehat{\phi}(f_k)\psi)_k$ converges weakly to $\widehat{\phi}(f)\psi$. To prove the norm-convergence it suffices[8] to check the convergence of the norm. By dominated convergence, we have:

$$\|\widehat{\phi}(f_k)\psi\|_{\mathcal{H}}^2 = \int |f_k|^2 \mathrm{d}\mu_\psi \to \int |f|^2 \mathrm{d}\mu_\psi = \|\widehat{\phi}(f)\psi\|_{\mathcal{H}}^2.$$

Let us now show the third point of the theorem.

Claim $\psi \in \mathrm{dom}(A)$ if and only if $\int x^2 \mathrm{d}\mu_\psi < +\infty$, in which case the integral coincides with $\|A\psi\|_{\mathcal{H}}^2$.

Proof of the claim. We consider the function $h_t(x) = (e^{itx} - 1)/t$ introduced in (1). By Stone theorem, we have $\psi \in \mathrm{dom}(A)$ if and only if $(\widehat{\phi}(h_t)\psi)_{t>0}$ has a well-defined limit in $\mathcal{H}$ as $t \to 0$.

For $M > 0$, observe that we have $\chi_{[-M,M]} \underset{M\to+\infty}{\to} 1$ in $\mathcal{B}(\mathbb{R})$, so $\widehat{\phi}(\chi_{[-M,M]})\psi$ converges to ψ. At fixed $M > 0$, observe that $x \mapsto x\chi_{[-M,M]}(x)$ is in $\mathcal{B}(\mathbb{R})$. In particular by strong convergence this sows that for all $\psi \in \mathcal{H}$, the element $\widehat{\phi}(\chi_{[-M,M]})\psi$ is in $\mathrm{dom}(A)$ and

$$A\widehat{\phi}(\chi_{[-M,M]})\psi = \widehat{\phi}(x\chi_{[-M,M]})\psi.$$

If $\int x^2 \mathrm{d}\mu_\psi$ is finite, then $(\widehat{\phi}(\chi_{[-M;M]})\psi, \widehat{\phi}(x\chi_{[-M;M]})\psi)_{M\in\mathbb{N}}$ is a Cauchy sequence. As the graph of A is closed, this implies that $\psi \in \mathrm{dom}(A)$ and $A\psi = \widehat{\phi}(x)\psi$ in the sense that $A\psi = \lim_{M\to+\infty} \widehat{\phi}(x\chi_{[-M;M]}\psi)$.

Conversely assume that $\psi \in \mathrm{dom}(A)$. Then on one hand we have:

$$\begin{aligned}\widehat{\phi}(x\chi_{[-M;M]})\psi &= \lim_{\substack{t\to 0\\ t\neq 0}} \widehat{\phi}(-ih_t)\widehat{\phi}(\chi_{[-M,M]})\psi \\ &= \lim_{\substack{t\to 0\\ t\neq 0}} \widehat{\phi}(\chi_{[-M,M]})\widehat{\phi}(-ih_t)\psi = \widehat{\phi}(\chi_{[-M,M]})A\psi.\end{aligned}$$

We get:

$$\int_{|x|\leq M} x^2 \mathrm{d}\mu_\psi(x) = \|\widehat{\phi}(x\chi_{[-M;M]})\psi\|_{\mathcal{H}}^2 \leq \|A\psi\|_{\mathcal{H}}^2.$$

By monotone convergence we get $\int x^2 \mathrm{d}\mu_\psi(x) < +\infty$, and its value is $\|A\psi\|_{\mathcal{H}}^2$ as

$$\underset{M\to+\infty}{\mathrm{s.}\lim} \widehat{\phi}(\chi_{[-M,M]})A\psi = A\psi.$$

Similarly if we take another sequence $(f_k)_k$ satisfying the third point of the theorem we obtain $\lim_{k\to+\infty} \widehat{\phi}(f_k)\psi = A\psi$ for $\psi \in \mathrm{dom}(A)$.

At last let us check the fifth point. That $\widehat{\phi}(f) \geq 0$ for $f \geq 0$ follows from the fact that then $\widehat{\phi}(f) = \widehat{\phi}(f^{1/2})^*\widehat{\phi}(f)$. If $\psi \in \mathrm{dom}(A)$ with $A\psi = \lambda\psi$, then $e^{itA}\psi = e^{it\lambda}\psi$ and from (3), we get $\widehat{\phi}(f)\psi = f(\lambda)\psi$ for f Schwartz. Following the proof, we obtain $\widehat{\phi}(f)\psi = f(\lambda)\psi$ for all $f \in \mathcal{B}(\mathbb{R})$. □

[8]Compute $\|\widehat{\phi}(f_k)\psi - \widehat{\phi}(f)\psi\|_{\mathcal{H}}^2$!

3. Spectral measures and the multiplication form

3.1. **Multiplication form of the spectral theorem.** In the previous section, we introduced the spectral measure μ_ψ associated to an element $\psi \in \mathcal{H}$.

Consider such a $\psi \in \mathcal{H}$, we define $\mathcal{H}_\psi$ as:

$$\mathcal{H}_\psi := \overline{\{f(A)\psi,\ f \in \mathcal{S}(\mathbb{R})\}}^{\|\cdot\|_{\mathcal{H}}}. \tag{6}$$

Definition 1. *We say that ψ is cyclic if $\mathcal{H}_\psi = \mathcal{H}$.*

As $\|f(A)\psi\|_{\mathcal{H}}^2 = \int |f|^2 \mathrm{d}\mu_\psi$, in particular $\|f_1(A)\psi - f_2(A)\psi\|_{\mathcal{H}}^2$ is equal to $\int |f-g|^2 \mathrm{d}\mu_\psi$, and $\mathcal{H}_\psi$ is isometric to $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ through the extension $U_\psi : \mathcal{H}_\psi \to L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ of the map:

$$\begin{array}{ccc} \{f(A)\psi,\ f \in \mathcal{S}(\mathbb{R})\} & \longrightarrow & L^2(\mathbb{R}, \mathrm{d}\mu_\psi), \\ f(A)\psi & \mapsto & f(x). \end{array}$$

Observe that for $f, g \in \mathcal{B}(\mathbb{R})$ we have:

$$\int |f|^2 \mathrm{d}\mu_{g(A)\psi} := \|f(A)g(A)\psi\|_{\mathcal{H}}^2 = \int |f|^2 |g|^2 \mathrm{d}\mu_\psi,$$

hence $\mathrm{d}\mu_{g(A)\psi} = |g|^2 \mathrm{d}\mu_\psi$. Recall $\psi \in \mathrm{dom}(A)$ *iff* $\int x^2 \mathrm{d}\mu_\psi$ is finite. Applying this result to elements in $\mathcal{H}_\psi$ and Stone theorem, we get that

(1) $\mathrm{dom}(A) \cap \mathcal{H}_\psi$ is dense in $\mathcal{H}_\psi$,
(2) the restriction of A to $\mathrm{dom}(A) \cap \mathcal{H}_\psi$ has range in $\mathcal{H}_\psi$ and is unitarily equivalent to the multiplication by x in $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$:

$$[U_\psi A U_\psi^{-1} f](x) = x f(x) \text{ for } f \in \mathcal{B}(\mathbb{R}) \text{ s.t. } \int x^2 |f(x)|^2 \mathrm{d}\mu_\psi < +\infty.$$

If ψ is cyclic this shows that up to the unitary U_ψ, A corresponds to the multiplication by x in some $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. If there is no cyclic vectors, but $\mathcal{H}$ is separable[9], then by a Gram-Schmidt procedure we obtain the following.

There exists a countable orthonormal family $(\psi_i)_{i \in I}$, such that the Hilbert space $\mathcal{H} = \overset{\perp}{\bigoplus}_{i \in I} \mathcal{H}_{\psi_i}$ is isometric to $\overset{\perp}{\bigoplus}_{i \in I} L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_i})$ through U. The conjugation of A by U is the multiplication by x in the L^2-space:

$$UAU^*(f_i(x))_{i \in I} = (x f_i(x))_{i \in I}, \text{ for } (f_i)_{i \in I} \text{ s.t. } \sum_{i \in I} \int x^2 |f_i(x)|^2 \mathrm{d}\mu_{\psi_i}(x) < +\infty.$$

This is the <u>multiplication form</u> of the spectral theorem. Using it, we easily show the following mapping property of the spectrum. **Exercise**.

Theorem 6. *[Spectral mapping] Let $f \in \mathcal{B}(\mathbb{R})$. Then we have the inclusion:*

$$\sigma(f(A)) \subset \overline{f(\sigma(A))}.$$

Furthermore the inclusion is an equality in the case where f is continuous or $\sigma(A)$ is compact.

Remark 7. *In particular we recover the fact for $f \in \mathcal{B}(\mathbb{R})$: if* supp $f = \{x \in \mathbb{R},\ f(x) \neq 0\}$ *does not intersect $\sigma(A)$, then $f(A) = 0$ (for instance observe that $\|f(A)\psi\|_{\mathcal{H}} = \||f|(A)\psi\|_{\mathcal{H}}$ and use the Weyl criterion for the spectrum on $|f|(A)$).*

[9] if not, then we must use Zorn's lemma, and deal with an uncountable family...

3.2. **Decomposition of the spectral measures.** We use the Lebesgue decomposition to decompose the spectral measures. We first recall the Radon-Nikodym theorem in the case we are interested in, that is the case of σ-finite [10] Borel measures on $\mathbb{R}$.

We give at the end of the section simple examples for this decomposition.

3.2.1. *The decomposition of a Borel measure.*

Definition 2. *Let μ, ν be two Borel measure.*
1. We say that μ is absolutely continuous w.r.t. ν and we write $\mu \ll \nu$ if for every Borel set Ω, $\nu(\Omega) = 0$ implies $\mu(\Omega) = 0$.
2. We say that μ is supported on the Borel set Ω if for every Borel set $O \in \mathrm{Bor}$, there holds:

$$\mu(O) = \mu(O \cap \Omega).$$

3. We say that μ and ν are mutually singular and we write $\mu \perp \nu$ if there exists a Borel set Ω such that

$$\mu(\Omega) = 0 \quad \& \quad \nu(\mathbb{R} \setminus \Omega) = 0.$$

Then the Radon-Nikodym theorem (for Borel measures) states the following.

Theorem 8. *Let μ, ν be two σ-finite Borel measure. The measure μ is absolutely continuous w.r.t. ν if and only if there exists a non-negative measurable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}\nu)$ defined ν-a.e. such that for all Borel set Ω, there holds:*

$$\mu(\Omega) = \int f \chi_\Omega \mathrm{d}\nu.$$

We also have the so-called Lebesgue decomposition of a Borel measure.

Theorem 9. *A σ-finite Borel measure μ can be uniquely decomposed as $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$, where μ_{ac} is absolutely continuous w.r.t. the Lebesgue measure λ, and $\mu_{\mathrm{sing}} \perp \lambda$.*

We may further decompose μ_{sing} by taking out the pure point part.

Definition 3. *A Borel measure μ is said to be a pure point measure if for every Borel set Ω, we have:*

$$\mu(\Omega) = \sum_{x \in \Omega} \mu(\{x\}).$$

Theorem 10. *A σ-finite Borel measure μ can be uniquely decomposed into three parts $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{cs}} + \mu_{\mathrm{pp}}$ where μ_{ac} is absolutely continuous w.r.t. the Lebesgue measureλ, μ_{pp} is a pure point measure and $\mu_{\mathrm{cs}} \perp \lambda$ and for every $x \in \Omega$, $\mu_{\mathrm{cs}}(\{x\}) = 0$.*

(μ_{cs} is called the singular continuous part of the measure: it is singular w.r.t. λ and is nevertheless continuous: it does not weight points.)

[10] In $\mathbb{R}$, it means that the measure of any compact set is finite.

3.2.2. *Decomposition of the spectrum.* For $\psi \in \mathcal{H}$, we thus decompose the spectral measure μ_ψ.

In the case where $\mathcal{H}$ is separable there exists a maximal spectral vector, such that for every $\phi \in \mathcal{H}$, we have $\mu_\phi \ll \mu_\psi$ (we say that μ_ψ is a maximal spectral measure). Indeed, using the multiplication form, and its associated family $(\psi_i)_{i\in I}$ with $I \subset \mathbb{N}$, it suffices to tale $\psi := \sum_{i\in I} 2^{-i}\psi_i$. In some sense μ_ψ contains all the information of the spectral decomposition.

The decomposition 10 gives rise to a decomposition of $\mathcal{H}$:

Definition 4. *For $\star \in \{\mathrm{ac}, \mathrm{pp}, \mathrm{cs}\}$, we define $\mathcal{H}_\star$ as follows:*

$$\mathcal{H}_\star := \left\{ \psi \in \mathcal{H},\ \mu_\psi \text{ is } \left\{ \begin{array}{ccc} \text{abs. cont. w.r.t. } \lambda & \text{if} & \star = \mathrm{ac}, \\ \text{cont. sing.} & \text{if} & \star = \mathrm{cs}, \\ \text{pure point} & \text{if} & \star = \mathrm{pp}\,. \end{array} \right. \right\}.$$

We also define the continuous part $\mathcal{H}_{\mathrm{cont}} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{cs}}$.

Claim This defines an orthogonal decomposition of $\mathcal{H}$ and for every $f \in \mathcal{B}(\mathbb{R})$, the $\mathcal{H}_\star$'s are stable through $f(A)$ and $A : \mathrm{dom}(A) \cap \mathcal{H}_\star \to \mathcal{H}_\star$.

We obtain at last the decomposition of the spectrum.

Definition 5. *For $\star \in \{\mathrm{ac}, \mathrm{pp}, \mathrm{cs}\}$, we define $\sigma_\star(A)$ as follows:*

$$\begin{array}{rcc} \sigma_{\mathrm{ac}}(A) & := & \sigma(A_{|\mathcal{H}_{\mathrm{ac}}}), \\ \sigma_{\mathrm{cs}}(A) & := & \sigma(A_{|\mathcal{H}_{\mathrm{cs}}}), \\ \sigma_{\mathrm{pp}}(A) & := & \{\text{eigenvalues of } A\}. \end{array}$$

We also call $\sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{cs}}(A)$ the continuous spectrum.

Due to this definition we have:

$$\sigma(A) = \sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{cs}}(A) \cup \overline{\sigma_{\mathrm{pp}}(A)}. \tag{7}$$

Proof of the claim. Thanks to the multiplication form, it suffices to prove it in $L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_0})$, where A corresponds to the multiplication by x. Let us drop the subscript ψ_0 in μ_{ψ_0} for short. Let $f, g \in L^2(\mathbb{R}, \mathrm{d}\mu)$ where $|f|^2\mathrm{d}\mu$ and $|g|^2\mathrm{d}\mu$ are mutually singular (this is the case when one is of one type ac, cs, pp and the other of another type). Then, we have:

$$|\langle U_{\psi_0}^{-1} f, U_{\psi_0}^{-1} g\rangle| = \left| \int \overline{f} g \mathrm{d}\mu \right| \leq \int |f||g|\mathrm{d}\mu.$$

By assumption, there exists $\Omega \in \mathrm{Bor}$ such that $\int_\Omega |f|^2\mathrm{d}\mu = \int_{\mathbb{R}\setminus\Omega} |g|^2\mathrm{d}\mu = 0$. By Cauchy-Schwarz inequality we have:

$$\begin{aligned} \int |f||g|\mathrm{d}\mu &= \int_\Omega |f||g|\mathrm{d}\mu + \int_{\mathbb{R}\setminus\Omega} |f||g|\mathrm{d}\mu, \\ &\leq \sum_{O \in \{\Omega, \mathbb{R}\setminus\Omega\}} \sqrt{\int_O |f|^2\mathrm{d}\mu \int_O |g|^2\mathrm{d}\mu} = 0. \end{aligned}$$

□

Remark 11. *If $\psi \in \mathcal{H}_{\rm pp} \setminus \{0\}$, then ψ is a linear combination of eigenfunctions of A. Indeed, consider μ_ψ: by assumption it is pure point and has finite mass. Thus there exists an almost countable family $(\lambda_i)_{i\in I}$ such that*

$$\mu_\psi(\mathbb{R}) = \sum_{i\in I} \mu_\psi(\{\lambda_i\}).$$

For every $i \in I$, the spectral measure of $\chi_{\{\lambda_i\}}(A)\psi$ is $\mu_\psi(\{\lambda_i\})\delta_{\lambda_i}$, hence $\chi_{\{\lambda_i\}}(A)\psi$ is an eigenfunction with eigenvalue λ_i. As

$$\lim_{\substack{J\to I\\ J\subset I}} \sum_{i\in K} \chi_{\{\lambda_i\}} = \chi_{\{\lambda_i, i\in I\}} \text{ in } \mathcal{B}(\mathbb{R}),$$

we have $\psi = \sum_{i\in I} \chi_{\{\lambda_i\}}(A)\psi$.

Simple examples.

(1) When $A = -\Delta$, then through the Fourier transform we see that $\sigma(-\Delta) = \sigma_{\rm ac}(-\Delta) = [0, +\infty)$.
(2) When A is compact, then $\sigma(A) = \overline{\sigma_{\rm pp}(A)} = \sigma_{\rm pp}(A) \cup \{0\}$.
(3) A s.a. compact perturbation of $-\Delta$ has the same absolutely continuous part plus an additional pure point part due to the perturbattion.
(4) An operator with continuous singular spectrum is not easy to exhibit. One can cheat and take $\mathcal{H} := L^2(\mathbb{R}, {\rm d}\mu_{\rm cs})$ where $\mu_{\rm cs}$ is continuous singular (*e.g.* supported on the Cantor set) and take for A the multiplication by x.

4. Projection-valued measure, discrete and essential spectrum

4.1. **Projection-valued measure.** If we restrict the functional calculus to the characteristic functions of Borel sets, we obtain a *projection-valued-measure*, in the sense that the function $\Omega \in \text{Bor} \mapsto P_\Omega := \chi_\Omega(A) \in \mathcal{L}(\mathcal{H})$ satisfies the following four points.

(1) For every Ω, P_Ω is an orthogonal projection (that is $P_\Omega^* = P_\Omega$ and $P_\Omega^2 = P_\Omega$).
(2) $P_\emptyset = 0$ and $P_\mathbb{R} = 1$.
(3) For a countable disjoint union of Borel sets $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$, that is $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$ we have:

$$P_\Omega = \operatorname*{s.lim}_{N\to+\infty} \sum_{n=1}^N P_{\Omega_n}.$$

(4) For two Borel sets Ω_1, Ω_2 there holds:

$$P_{\Omega_1\cap\Omega_2} = P_{\Omega_1} P_{\Omega_2}.$$

In particular, given $\psi \in \mathcal{H}$, then

$$\Omega \in \text{Bor} \mapsto \langle \psi, P_\Omega \psi\rangle$$

defines a measure in the original sense. We can integrate a bounded Borel function f w.r.t. that measure which we denote by

$$\int f(\lambda){\rm d}\langle \psi, P_\lambda\psi\rangle.$$

Conversely a projection valued measure defines a self-adjoint operator in a unique: we refer the reader to the references. This is the p.v.m. form of the spectral theorem.

Theorem 12. *There is a bijection between self-adjoint operators and projection valued measures. Given a projection valued measure P_Ω its associated self-adjoint operator A is defined by:*

$$\langle \psi, f(A)\psi\rangle = \int f(\lambda)\mathrm{d}\langle \psi, P_\lambda \psi\rangle.$$

Remark 13. *We obtain $\langle \phi, f(A)\psi\rangle$ by polarization as done in Section 2.*

4.2. **Discrete and essential spectrum.**

Definition 6. *We define the discrete spectrum $\sigma_{\mathrm{disc}}(A)$ as the set*

$$\sigma_{\mathrm{disc}}(A) = \{\lambda \in \sigma(A),\ \exists \varepsilon > 0,\ \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \text{ is finite dimensional}\}.$$

The essential spectrum is its complement:

$$\sigma_{\mathrm{ess}}(A) = \{\lambda \in \sigma(A),\ \forall \varepsilon > 0,\ \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \text{ is infinite dimensional}\}.$$

In other words, $\sigma_{\mathrm{disc}}(A)$ corresponds to the set of eigenvalues with finite multiplicity and which are isolated in the spectrum.

The essential spectrum corresponds to the union of 1. the continuous spectrum 2. the set of limit points of eigenvalues 3. the set of eigenvalues with infinite multiplicity.

Almost by definition, we obtain the Weyl criterion for the essential spectrum.

Theorem 14. *There holds $\lambda \in \sigma_{\mathrm{ess}}(A)$ if and only if there exists a sequence of $\|\cdot\|_{\mathcal{H}}$-normalized elements $(\psi_n)_n$ in $\mathrm{dom}(A)$ such that $\|(A-\lambda)\psi_n\|_{\mathcal{H}} \to 0$ and $\langle \psi_n, \psi_m\rangle = \delta_{nm}$ (that is it is an orthonormal family).*

Remark 15. ***Exercise: show that you can replace the condition $\langle \psi_n, \psi_m\rangle = \delta_{nm}$ by the requirement that $(\psi_n)_n$ converges weakly to 0.***

In some sense $\sigma_{\mathrm{ess}}(A)$ is the part of the spectrum which is stable under "small" perturbations. We have indeed[11].

Theorem 16. *[Stability of the essential spectrum] Let A and B s.a. on $\mathcal{H}$. Assume that there exists $z \in \rho(A) \cap \rho(B)$ such that $R_A(z) - R_B(z)$ is compact. We recall that $R_C(z) = (C-z)^{-1}$, for $C = A$ and $C = B$.*

Then $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$.

Remark 17. *The condition holds when B is a compact perturbation of A.*

Proof. We use the Weyl criterion for the essential spectrum and establishes the double inclusion. By symmetry (of the role played by A and B), it suffices to show that $\sigma_{\mathrm{ess}}(B) \subset \sigma_{\mathrm{ess}}(A)$.

Let $\lambda \in \sigma_{\mathrm{ess}}(A)$ and (ψ_n) be a Weyl sequence for λ (Theorem 14). We have:

$$\big(R_A(z) - (\lambda - z)^{-1}\big)\psi_n = \frac{R_A(z)}{z-\lambda}(A-\lambda)\psi_n,$$

[11]It can be generalized, see for instance the book of Reed and Simon.

hence $\|\big(R_A(z)-(\lambda-z)^{-1}\big)\psi_n\|_{\mathcal{H}} \underset{n\to\infty}{\to} 0$ and $\big|\|R_A(z)\psi_n\|_{\mathcal{H}}-|\lambda-z|^{-1}\big| \underset{n\to\infty}{\to} 0$.

As[12] $\psi_n \rightharpoonup_{\mathcal{H}} 0$ and $R_A(z)-R_B(z)$ compact, then $\|(R_A(z)-R_B(z))\psi_n\|_{\mathcal{H}}$ converges to 0 and $\|\big(R_B(z)-(\lambda-z)^{-1}\big)\psi_n\|_{\mathcal{H}} \underset{n\to\infty}{\to} 0$.

Let $\phi_n := R_B(z)\psi_n \in \mathrm{dom}(B)$. The following three points hold.

(1) $\phi_n \rightharpoonup_{\mathcal{H}} 0$ as $\psi_n \rightharpoonup_{\mathcal{H}} 0$ and $R_B(z)$ is bounded,

(2) there holds

$$\|\phi_n\|_{\mathcal{H}} = \|(R_B(z)-R_A(z))\psi_n + R_A(z)\psi_n\|_{\mathcal{H}} \underset{n\to\infty}{\to} |\lambda-z|^{-1},$$

(3) there holds

$$\begin{aligned}\|(B-\lambda)\phi_n\|_{\mathcal{H}} = \|(B-z)\phi_n + (z-\lambda)\phi_n\|_{\mathcal{H}} &= \|\psi_n + \tfrac{z-\lambda}{B-z}\psi_n\|_{\mathcal{H}},\\ &= \|\psi_n + \tfrac{z-\lambda}{A-z}\psi_n + (z-\lambda)(R_B(z)-R_A(z))\psi_n\|_{\mathcal{H}},\\ &\underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Up to normalizing ϕ_n and using Remark 15, we obtain a Weyl sequence for B.

□

References

[1] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.

[2] Gerald Teschl, *Mathematical methods in quantum mechanics*, 2nd ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. With applications to Schrödinger operators.

[12] Exercise: why? Use a Hilbert basis!