TENSOR PRODUCTS, BOSONS & FERMIONS

DEF: (Tensor product of Hilbert spaces)

Let $\mathcal{H}_1, \dots \mathcal{H}_N$ be Hilbert spaces, let $\varphi_i \in \mathcal{H}_i$, $i = 1, \dots N$.

Then we define a multisemilinear form

$$\varphi_1 \otimes \dots \otimes \varphi_N : \mathcal{H}_1 \times \dots \times \mathcal{H}_N \longrightarrow \mathbb{C}$$

by $\varphi_1 \otimes \dots \otimes \varphi_N (f_1, \dots, f_N) := \prod_{i=1}^{N} \langle f_i, \varphi_i \rangle_{\mathcal{H}_i}$.

Let $D_N = \{$finite linear combinations of such $\varphi_1 \otimes \dots \otimes \varphi_N\}$.

called "elementary tensors"

Define a scalar product on D_N by linear extension of

$$(*) \qquad \langle \varphi_1 \otimes \dots \otimes \varphi_N, \psi_1 \otimes \dots \otimes \psi_N \rangle := \prod_{i=1}^{N} \langle \varphi_i, \psi_i \rangle_{\mathcal{H}_i}.$$

$\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N := \overline{D_N}$, where the closure is w.r.t. the norm induced by $(*)$.

PRP.: If $(\varphi_i^{(k)})_{i \in \mathbb{N}}$ are ONBs of $\mathcal{H}_k$, then

$$\{\varphi_{i_1}^{(1)} \otimes \dots \otimes \varphi_{i_N}^{(N)} : i_1, \dots i_N \in \mathbb{N}\}$$

is an ONB of $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$.

PROOF: Reed & Simon Vol. 1, p.50. $\blacksquare$

LEMMA: Consider $L^2(\mathbb{R}^d)$ with ONB $(e_i)_{i\in\mathbb{N}}$.

Then there exists a unitary map

$$U: \underbrace{L^2(\mathbb{R}^d)\otimes \cdots \otimes L^2(\mathbb{R}^d)}_{N \text{ factors}} \longrightarrow L^2(\mathbb{R}^{dN}),$$

namely

$$\left(U e_{i_1}\otimes\cdots\otimes e_{i_N}\right)(x_1,\dots x_N) = e_{i_1}(x_1)\, e_{i_2}(x_2)\cdots e_{i_N}(x_N).$$

PROOF: Calculate the scalar product. ∎

LEMMA: If $\mathcal{H}$ is a Hilbert space, then there is a unitary

$$U: L^2(\mathbb{R}^d)\otimes\mathcal{H} \longrightarrow L^2(\mathbb{R}^d,\mathcal{H}),$$

space of $\mathcal{H}$-valued functions

namely, for $f\in L^2(\mathbb{R}^d)$, $e\in\mathcal{H}$:

$$U(f\otimes e)(x) = \underbrace{\underbrace{f(x)}_{\in\mathbb{C}}\, e}_{\in\mathcal{H}} \quad \text{almost every } x\in\mathbb{R}^d.$$

PROOF:

$$\|U(f\otimes e)\|^2_{L^2(\mathbb{R}^d,\mathcal{H})} \overset{\text{def}}{=} \int \|U(f\otimes e)(x)\|^2_{\mathcal{H}}$$

$$= \int dx\, |f(x)|^2\, \|e\|^2_{\mathcal{H}} = \|f\|^2_{L^2(\mathbb{R}^d)}\, \|e\|^2_{\mathcal{H}}$$

$$= \|f\otimes e\|^2_{L^2(\mathbb{R}^d)\otimes\mathcal{H}}.$$

recall (*) from first page.

For scalar products use polarization. ∎

DEF: (algebraic tensor product of operators)

let $A: D(A) \subset \mathcal{H}_1 \to \mathcal{H}_1$, $B: D(B) \subset \mathcal{H}_2 \to \mathcal{H}_2$ be densely defined operators,

let $D(A) \otimes D(B) := \{$ finite linear combinations of elementary tensors $\varphi_1 \otimes \varphi_2$, $\varphi_1 \in D(A)$, $\varphi_2 \in D(B)\}$.

Define $(A \otimes_{alg} B)(\varphi_1 \otimes \varphi_2) := (A\varphi_1) \otimes (A\varphi_2)$ as an operator with domain $D(A) \otimes D(B)$.

LEMMA: (i) $A \otimes_{alg} B$ is well-defined.

(ii) $A^* \otimes_{alg} B^* \subset (A \otimes B)^*$.

(iii) A, B are closable $\Rightarrow$ $A \otimes_{alg} B$ is closable.

(iv) A, B are symmetric $\Rightarrow$ $A \otimes_{alg} B$ is symmetric.

PROOF: (i) have to show that for

$$\gamma = \sum_i \alpha_i \varphi_i \otimes \psi_i, \quad \gamma = \sum_i \tilde{\alpha}_i \tilde{\varphi}_i \otimes \tilde{\psi}_i,$$

$(A \otimes_{alg} B)\gamma$ is defined indep. of the choice of decomposition. (do the calc!)

(ii) For $\gamma = \sum_i \alpha_i \varphi_i \otimes \psi_i$, $\omega = \sum_i \beta_i \omega_i \otimes \xi_i \in D(A^*) \otimes D(B^*)$:

$$\langle \omega, (A \otimes_{alg} B)\gamma \rangle$$

$$= \langle \omega, \sum_i \alpha_i \, A e_i \otimes B \psi_i \rangle$$

$$= \sum_j \sum_i \overline{\beta_j} \alpha_i \langle \omega_j, A e_i \rangle \langle \xi_j, B \psi_i \rangle$$

$$= \sum_j \sum_i \overline{\beta_j} \alpha_i \langle A^* \omega_j, e_i \rangle \langle B^* \xi_j, \psi_i \rangle$$

$$= \langle (A^* \otimes_{alg} B^*) \omega, \gamma \rangle .$$

(iii), (iv): exercises. ∎

DEF: If A, B are closable operators, then

$$A \otimes B := \overline{A \otimes_{alg} B} .$$

(**) Furthermore $A + B := \overline{A \otimes \mathbb{1} + \mathbb{1} \otimes B}$ on $D(A) \otimes D(B)$.

LEMMA: If A, B are bounded, then $\|A \otimes B\| \leq \|A\| \, \|B\|$.

PROOF: Exercise again. ∎

REMARK: The operator $H = \sum_{k=1}^{N} -\Delta_{x_k} + V(x_k)$ on $L^2(\mathbb{R}^{3N})$ can also be understood as an operator on $\bigotimes_{i=1}^{N} L^2(\mathbb{R}^3)$ (recall the unitary equivalence to $L^2(\mathbb{R}^{3N})$) if we read it in the sense of (**).

THM: Let A, B essentially self-adjoint. Then $A \otimes B$ and $A + B$ are essentially self-adjoint on $D(A) \otimes D(B)$, and

$$\sigma(A \otimes B) = \overline{\sigma(\bar{A})\sigma(\bar{B})}$$

$$\sigma(A + B) = \overline{\sigma(\bar{A}) + \sigma(\bar{B})}.$$

PROOF: Reed & Simon Thm. VIII.33 and its corollary. $\blacksquare$

FERMIONS & BOSONS → see [Ballentine], Chapters 17.1–17.3

A quantum system of N particles has Hilbert space

$$\bigotimes_{i=1}^{N} L^2(\mathbb{R}^3) \simeq L^2(\mathbb{R}^{3N}).$$

Identical particles are indistinguishable by measurements!

I.e. two electrons (same spin, same mass, same charge) cannot be told apart.

This phenomena is <u>not</u> quantum, it appears already in classical statistical mechanics.

(Think of the following: put 2 identical balls in a box, close, come back later and open — you cannot tell which ball is which!)

Now let P_σ the representation of permutations $\sigma \in S_N$:

$$(P_\sigma \psi)(x_1, \dots x_N) := \psi(x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(N)}).$$

DEF: If $A = A^*$ on $L^2(\mathbb{R}^{3N})$:

A is an <u>observable</u> if and only if

$$(*) \qquad \langle P_\sigma \psi, A P_\sigma \psi \rangle = \langle \psi, A\psi \rangle \quad \forall \psi \in L^2(\mathbb{R}^{3N}),\ \forall \sigma \in S_N.$$

REMARKS:

- Other operators appear in calculations, but they do not correspond to measurements.

- The above could also be seen as a def. of particles being identical.

- It is often claimed that indistinguishability relates to Heisenberg indeterminacy. This is not true! Every wave function satisfies indeterminacy.

OBSERVATION: For 3 particles, consider the decomp. into subspaces which are invariant under the action of all permutations:

$$\bigotimes_{i=1}^{3} L^2(\mathbb{R}^3) = \mathcal{H}_{symm} \oplus \mathcal{H}_{antisymm} \oplus \mathcal{H}_I \oplus \mathcal{H}_{II}.$$

i.e. $\psi \in \mathcal{H}_{symm.} \Leftrightarrow P_\sigma \psi = \psi \quad \forall \sigma \in S_N$,
$\psi \in \mathcal{H}_{antisymm} \Leftrightarrow P_\sigma \psi = \mathrm{sgn}(\sigma)\, \psi \quad \forall \sigma \in S_N$,

and $P_\sigma \mathcal{H}_I \subset \mathcal{H}_I$, $P_\sigma \mathcal{H}_{II} \subset \mathcal{H}_{II}$,
but generally $P_\sigma \psi \neq \text{const.}\, \psi$ for $\psi \in \mathcal{H}_I$ or $\mathcal{H}_{II}$.

POSTULATE: In dimension $d \geq 3$ there exist only <u>fermions</u> and <u>bosons</u>, i.e. for a given system of identical particles either all wave functions are in $\mathcal{H}_{antisymm.}$, or all wave functions are in $\mathcal{H}_{symm}$.

RMK: - (*) implies that no linear combinations between different subsectors are observable.
- (*) implies that $\psi_0 \in \mathcal{H}_{symm} \Rightarrow \psi_t \in \mathcal{H}_{symm}$,

since the Hamiltonian H is an observable.

In the same way $\Psi_0 \in \mathcal{H}_{antisymm} \Rightarrow \Psi_t \in \mathcal{H}_{antisymm}$.

- the fact that $\Psi \notin \mathcal{H}_I$ and $\Psi \notin \mathcal{H}_{II}$ is due to experimental observation!

REMARK: In relativistic QM it is a rigorous theorem that:

- particles with integer spin are bosons, i.e. $\Psi \in \mathcal{H}_{symm}$.
- particles with spin $\in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ are fermions, i.e. $\Psi \in \mathcal{H}_{antisymm}$.
- there are no other spin values.

But this is a property of representations of the Poincaré group, and does <u>not</u> hold in our non-relativistic theory!