Many-Body Bose Gases and the Bogoliubov Approximation

A particle one-body Hilbert Space $\mathfrak{h}$

N-Body space $\bigotimes^N \mathfrak{h} = \mathcal{H}_N$

Bosonic subspace $\mathcal{H}_N^B = \bigotimes^N_{Sym} \mathfrak{h} \subseteq \mathcal{H}_N$, $\mathcal{H}_N^{Fermi} = \bigwedge^N \mathfrak{h}$

Projection $P_N^B : \mathcal{H}_N \to \mathcal{H}_N^B$. $\quad P_N^B = \frac{1}{N!} \sum_{\sigma \in S_N} U_\sigma$

Hamiltonian 2-Body interactions:

$$H_N = \sum_{i=1}^N T_i + \sum_{1 \le i < j \le N} W_{ij} \qquad T : D(T) \subseteq \mathfrak{h} \to \mathfrak{h}$$

$$P_2^B W = W P_2^B \qquad W : D(W) \subseteq \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$$

Fock Space: $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^B \qquad H = \bigoplus_{N=0}^{\infty} H_N$

Ground State energy

$$E = \inf_{\substack{\|\psi\|=1 \\ \psi \in D(H)}} \langle \psi | H \psi \rangle \qquad E_N = \inf_{\substack{\psi \in D(H_N) \\ \|\psi\|=1}} \langle \psi | H_N \psi \rangle$$

$$= \inf_N E_N$$

Free energy at temp T

$$F(T) = \inf \left(\mathrm{Tr}\,[\Gamma H] - T S(\Gamma) \right)$$

$\Gamma : \mathcal{F} \to \mathcal{F}$, $0 \le \Gamma$, $\mathrm{Tr}\,\Gamma = 1$ $\quad S(\Gamma) = -\mathrm{Tr}\,\Gamma \ln \Gamma$

Ground state of Free Bose Gas: $W=0$.

$E_N = N \inf \operatorname{spec}(T)$ $\qquad \|u\|=1 \quad Tu = \lambda_0 u$

$= N\lambda_0$ $\qquad$ Ground State $\Psi = \bigotimes^N u$

Condensate.

Example $\quad \mathfrak{h} = L^2(\Lambda) \qquad \Lambda = [0,L]^3$

$T = -\Delta - \mu$, μ chemical potential,

$-\Delta$ periodic Boundary conditions.

$W = V(x-y)$ $\quad V: \mathbb{R}^3 \to \mathbb{R}$ compact support.

potential.

Energy density: $e = \lim_{L\to\infty} \frac{E_\Lambda}{L^3}$, $f = \lim_{L\to\infty} \frac{F_\Lambda}{L^3}$.

2nd quantization

If $u \in \mathcal{h}$ define

$$a^\dagger(u) : \mathcal{H}_N^B \to \mathcal{H}_{N+1}^B \qquad a^\dagger(u)\Psi = \sqrt{N+1}\, P_{N+1}^B (u \otimes \Psi)$$

$$a(u) = a^\dagger(u)^* : \mathcal{H}_{N+1}^B \to \mathcal{H}_N^B$$

Canonical Commutation Relations:

$$[a(u), a^\dagger(v)] = (u,v)_{\mathcal{h}}\, I$$

$a^\dagger$ = creation operator $\quad a$ = annihilation operator

$$[a(u), a(v)] = [a^\dagger(u), a^\dagger(v)] = 0$$

The Hamiltonian in 2^{nd} quantization:

Let $u_1, u_2, \ldots$ be an orthonormal basis for $\mathfrak{h}$

$(u_i \in D(T), \ u_i \otimes u_j \in D(W))$ $\qquad H_N = \sum_{i=1}^{N} T_i + \sum_{1 \le i < j \le N} W_{ij}$

$$H = \bigoplus_{N=0}^{\infty} H_N = \sum_{i,j} \langle u_i, T u_j \rangle a^+(u_i) a(u_j)$$

$$+ \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} \langle u_\alpha \otimes u_\beta, W u_\mu \otimes u_\nu \rangle a^+(u_\alpha) a^+(u_\beta) a(u_\mu) a(u_\nu)$$

Particle number operator $\mathcal{N} = \bigoplus_{N=0}^{\infty} N \cdot I_{\mathcal{H}_N}$

$$= \sum_i a^+(u_i) a(u_i)$$

The translation invariant spectrum $\qquad |\Lambda| = L^3$

$$a^+_p = a^+\left(\frac{e^{-ixp}}{\sqrt{|\Lambda|}}\right) \qquad p \in \Lambda^*$$

$$\sum_{p\in\Lambda^*} (p^2-\mu)\, a^+_p a_p + \frac{1}{2}\frac{1}{|\Lambda|}\sum_{k,p,q} \hat{V}(k)\, a^+_{p+k} a^+_{q-k} a_q a_p$$

The Bogolubov approximation

a la Bogolubov:

- Assumption: close to a condensate, i.e. $\exists u_0 \in \mathcal{h}$
$$\langle a^+(u_0) a(u_0)\rangle \approx \langle \mathcal{N}\rangle$$
- Replace $a^+(u_0)$ by a number (c-number substitution)

Note the Hamiltonian no longer preserves particle no.

- Ignore terms with 3 or 4 $a^+(u_i)$ or $a(u_i)$ with $i \neq 0$.

Translation invariant case: $a^+(u_0) = a_0^+ \to \sqrt{\rho_0 |\Lambda|}$

ρ_0 = density in the condensate.

Bogolubov Hamiltonian

$$H_{Bog} = \sum_{p \neq 0} (p^2 - \mu) a_p^+ a_p - \rho_0 |\Lambda| \mu + \frac{1}{2} \rho_0^2 \hat{V}(0) |\Lambda|$$

$$+ \frac{1}{2} \rho_0 \sum_{p \neq 0} \hat{V}(p) (a_p^+ a_{-p}^+ + a_p a_{-p} + a_p^+ a_p + a_{-p}^+ a_{-p})$$

$$+ \rho_0 \hat{V}(0) \sum_p a_p^+ a_p ,$$

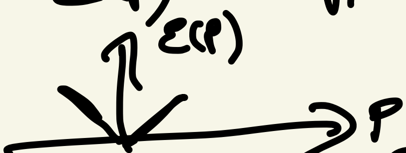
The quadratic Hamiltonian can be "diagonalized by" a Bogoliubov transformation.

$$d_p^+ = \alpha_p a^+_p + \beta_p a_{-p} \qquad \alpha_p^2 - \beta_p^2 = 1$$

$$[d_p, d_q^+] = \delta_{pq} \qquad [d_p, d_q] = [d_p^+, d_q^+] = 0$$

$$H_{\text{Bogoliubov}} = \sum_p \varepsilon(p)\, d_p^+ d_p + E + \frac{1}{2}\hat{V}(0)\,\rho_0^2\,|\Lambda|$$

$$\varepsilon(p) = 2\sqrt{A(p)^2 - B(p)^2}\,, \quad E = \sum_{p\neq 0} \sqrt{A_p^2 - B_p^2} - A(p)$$



$$A(p) = \frac{1}{2}\Big[p^2 \underbrace{-\mu + \rho_0 \hat{V}(0)}_{\approx 0} + \rho_0 \hat{V}(p)\Big]\,, \quad B(p) = \rho \hat{V}(p)$$

The variational approach.

$\Gamma = Z^{-1} e^{-\beta \sum \varepsilon(p) a_p^+ a_p dp}$ $\quad \mathrm{Tr}\,\Gamma = 1$ Gaussian or quasi-free

Such states satisfy Wick' Thm. $\langle \cdot \rangle = \mathrm{Tr}[\Gamma \,\cdot\,]$.

$$\langle a_1^+ a_2^+ a_3 a_4 \rangle = \langle a_1^+ a_4 \rangle \langle a_2^+ a_3 \rangle + \langle a_1^+ a_3 \rangle \langle a_2^+ a_4 \rangle + \langle a_1^+ a_2^+ \rangle \langle a_3 a_4 \rangle$$

Everything is determined by

$$\gamma(p) = \langle a_p^+ a_p \rangle \qquad \alpha(p) = \langle a_p^+ a_{-p}^+ \rangle = \langle a_p a_{-p} \rangle$$

They satisfy:

$$\alpha(p)^2 \leq \gamma(p)(\gamma(p)+1) \Leftarrow \langle (a_p^+ + t a_{-p})(a_p + t a_{-p}^+) \rangle \geq 0 \quad \forall t$$

c- number substitution:

Unitary: $U_{\rho_0}: \mathcal{F} \to \mathcal{F}$ $\qquad U_{\rho_0}^* a_p^+ U_{\rho_0} = a_p^+ + \sqrt{\rho_0 |\Lambda|}\, \delta_{p0}$

Energy functional (upper bound to the energy)

$$\mathcal{E}(\gamma, \alpha, \rho_0) = \langle U_{\rho_0}^* H U_{\rho_0} \rangle_{\gamma,\alpha} \quad \Big| \; \rho = \rho_0 + \frac{1}{|\Lambda|} \sum_p \gamma(p)$$

$$= \sum_{p \neq 0} p^2 \gamma(p) - \mu \rho |\Lambda| + \frac{1}{2} \hat{V}(0) \rho^2$$

$$+ \sum_{p \neq 0} \rho_0 \hat{V}(p) \big(\gamma(p) + \alpha(p)\big)$$

$$+ \frac{1}{2|\Lambda|} \sum_p \sum_q \hat{V}(p-q) \big[\gamma(p)\gamma(q) + \alpha(p)\alpha(q)\big]$$

Entropy $\mathrm{Tr}\left[-\Gamma \ln \Gamma\right] = \sum_p \left(\beta(p)+\frac{1}{2}\right)\ln\left(\beta(p)+\frac{1}{2}\right)$

$$- \left(\beta(p)-\frac{1}{2}\right)\ln\left(\beta(p)-\frac{1}{2}\right)$$

$$\beta(p) = \sqrt{\left(\gamma(p)+\frac{1}{2}\right)^2 - \alpha(p)^2} \geq \frac{1}{2}$$

$$\Uparrow$$

$$\alpha(p)^2 \leq \gamma(p)\left(\gamma(p)+1\right)$$

$$\frac{1}{|\Lambda|}\sum_p \xrightarrow[L\to\infty]{} (2\pi)^{-3}\int dp$$