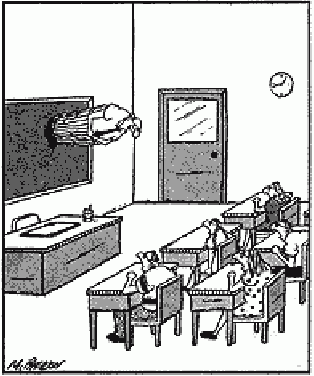

"Good morning, and welcome to
The Wonders of Physics."

Hopf term and Anyons

Topological Excitations II

Advanced Seminar Quantum Field Theory of Low-dimensional Systems

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Overview

1. Motivation and homotopy theory
   - Spin in different space dimensions
   - Repetition: Homotopy groups
2. Hopf term in the non-linear sigma model
   - Repetition: the O(3) non-linear sigma model
   - Introducing the Hopf term
   - Connection of linking number and Hopf term
   - The topological action for the sigma model
3. Realization of anyons
   - Continuity of pair creation
   - Spin and statistics of skyrmions
   - Remark: Skyrmions in 3+1 dimensions
4. Summary

Overview

Spin in different space dimensions

- In 3+1 dimensions: 3 axes of rotation $\rightsquigarrow$ 3 operators of angular momentum with commutator relations

$$[S_i, S_j] = i\varepsilon_{ijk} S_k.$$

$\rightsquigarrow$ Spin is integer or half-integer, eigenvalues:

$$\mathbf{S}^2 \left|s, m\right\rangle = s(s+1) \left|s, m\right\rangle , \quad \text{with } s \in \frac{1}{2}\mathbb{N}.$$

- In 2+1 dimensions: Only *one* axis of rotation exists.
$\rightsquigarrow$ only *one* operator of angular momentum, *no* commutators!

Result

In 2+1 dimensions, spin is not restricted to integer and half-integer values.

Homotopy

Connection with quantum statistics?
Idea: "coarse" classification of mappings allows us to discuss interchange of particles etc.

Definition (Continuous deformation/homotopy)

Let X be a topological space.
A homotopy between two continuous mappings $f_1, f_2 : S^n \to X$ is a continuous mapping $h : S^n \times [0,1] \to X$ with

$$h(x,0) = f_1(x),\ h(x,1) = f_2(x) \quad \forall x \in S^n.$$

Define $f_1 \sim f_2$ (equivalence) if a homotopy between them exists.

Homotopy groups

Fix a base point $x_0 \in X$, $s_0 \in S^n$ and require

$$f(s_0) = x_0, \quad h(s_0, t) = x_0 \quad \forall t \in [0, 1].$$

Notation: $C(S^n, X) := \{f : S^n \to X \mid f \text{ continuous}; \; x_0, s_0 \text{ fixed}\}$.

Definition (Homotopy groups)

The n^{th} homotopy group is the set of equivalence classes

$$\pi_n(X) := C(S^n, X)/\sim .$$

Remark: $\pi_n(X)$ does not depend on x_0 and s_0 (if X is path-connected), but they need to be fixed!

The multiplication law in $\pi_n(X)$

n-cube $I^n := [0,1] \times \cdots \times [0,1]$, surface (boundary) ∂I^n.

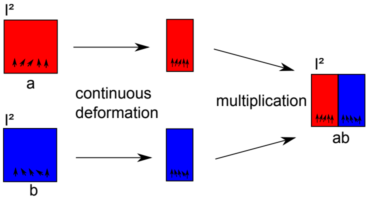
$$\begin{array}{lll} \text{Identify mappings} & S^n \to X, & s_0 \mapsto x_0 \\ \text{with mappings} & I^n \to X, & \partial I^n \mapsto x_0. \end{array} \quad ①$$

For two such mappings α, β:

$$\alpha \cdot \beta(x_1, \ldots x_n) := \begin{cases} \alpha(2x_1, x_2, \ldots x_n) & 0 \leq x_1 \leq 1/2 \\ \beta(2x_1 - 1, x_2, \ldots x_n) & 1/2 < x_1 \leq 1 \end{cases}$$

Example 1:
mappings $a, b : S^2 \to S^2$:

Example 2: $\pi_1(X)$,
i.e. $S^1 \to X$ ②



Homotopy groups of spheres

Example: Take $X = S^n$. $\pi_k(X) = \pi_k(S^n) \cong$?

Homotopy groups of spheres

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en.wikipedia.org/wiki/Homotopy_groups_of_spheres:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
S^0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} = (\{0, 1, \ldots m-1\}, + \mod m)$.

Overview

The non-linear sigma model

- Describes a *continuous* spin field in the 2d plane:

$$n : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3, \quad x \mapsto n(x), \text{ a unit vector.}$$

- Energy given by classical Hamiltonian:

$$E(n) = \int \sum_{a=1}^{3} | \underbrace{\nabla n^a}_{\text{spatial derivatives}} |^2 \, \mathrm{d}^2 x \geq 0$$

- No prefered orientation of n: degenerate ground state, e.g. $n = (1, 0, 0)$, which has $\partial_i n^a = 0$, $\rightsquigarrow E(n) = 0$.

Ground state and excitations

Excitations: $E(n) < \infty$ requires "rapid decrease" of $\partial_i n^a$.
$\rightsquigarrow$ use boundary condition

$$n(x) \to (1,0,0) \quad (|x| \to \infty).$$

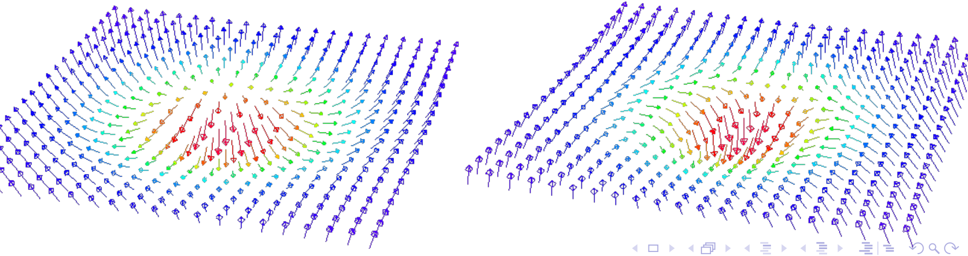
So all those field configurations n can be seen as continuous mappings $S^2 \to S^2$:

$$n : \underbrace{\mathbb{R}^2 \cup \{\infty\} \cong S^2}_{\text{compactified position space}} \to \underbrace{S^2}_{\text{internal space (spin)}}$$

$$x \in \mathbb{R}^2 \mapsto n(x), \quad \infty \mapsto (1,0,0)$$

Solitons: topological excitations

- Continuous mappings $S^2 \to S^2$ can be classified in homotopy classes, elements of $\pi_2(S^2)$.
- $\pi_2(S^2) \cong \mathbb{Z}$ with isomorphism $\phi : n \mapsto Q[n]$,
  Q is the topological charge/Pontryagin number/winding number.
- Q is a homotopy invariant, i.e. Q is invariant under continuous deformation.
- Time evolution is a continuous deformation of the field, so $Q[n]$ does not change with time.
- *Skyrmion*: a field configuration with $Q = 1$. *Antiskyrmion*: $Q = -1$.



Inclusion of time dependence

- Configuration space of the sigma model:

$$X = \left\{ n : \mathbb{R}^2 \to S^2 \mid n \text{ is continuous, } n(x) \to (1,0,0) \ \ (|x| \to \infty) \right\}.$$

- Paths $\underline{n} : t \mapsto \underline{n}(t)$ in X parameterized by time t:
  Choose boundary condition:
  $t_0 = \infty, -\infty$ in time $\mapsto s_0 = n_{\text{ground}} \equiv (1,0,0) \in X$.
- $\rightsquigarrow$ Every such closed path in X is

$$\underline{n} : \mathbb{R}_t \times \mathbb{R}^2 \to S^2, \quad (t,x) \mapsto \underline{n}(t,x)$$

  with

$$\underline{n}(t,x) \to (1,0,0) \quad \text{for } |\underbrace{(t,x)}_{\in \mathbb{R}^3}| \to \infty.$$

Inclusion of time dependence

By compactification $\mathbb{R}^3 \cup \{\infty\} \cong S^3$:
field evolution $\underline{n}$ can be seen as continuous mapping

$$\underline{n} : S^3 \to S^2.$$

Result

Every path $\underline{n}$ which

- *includes only finite energy configurations*
- *and has the ground state at times $t = \pm\infty$*

represents an element of $\pi_3(S^2)$.

The Hopf term

Let $\underline{n} : S^3 \to S^2$.
There exists a mapping H ("Hopf term") with the properties:

- $H[\underline{n}] \in \mathbb{Z}$
- H is a homotopy invariant, i.e. it does not change under continuous deformation of $\underline{n}$.
- $H : \pi_3(S^2) \to \mathbb{Z}$ is a homomorphism:

$$H[\underline{n}_1 \cdot \underline{n}_2] = H[\underline{n}_1] + H[\underline{n}_2].$$

- For calculation of $H[\underline{n}]$, use the linking number:

Connection of linking number and Hopf term

Lemma (Sard's theorem)

Let $\underline{n} : S^3 \to S^2$. Then (almost) every point in S^2 will have as its inverse image in S^3 a collection of closed curves.

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Theorem (Linking number)

Let $\underline{n} : S^3 \to S^2$. Choose two values:

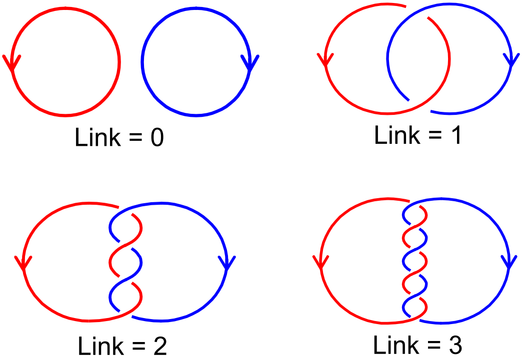
$$\underline{n}(x_a, t_a), \ \underline{n}(x_b, t_b) \in S^2.$$

Their "worldlines" in $\mathbb{R}_t \times \mathbb{R}^2$ are two collections of closed curves: γ_a and γ_b and

$$H[\underline{n}] = Link(\gamma_a, \gamma_b),$$

where $Link$ is the linking number of the curves:

Linking number



The topological action

Add Hopf term to the action of the sigma model
(existence and value of s to be decided on microscopic level):

$$S[\underline{n}] := \underbrace{\int \mathrm{d}t\mathrm{d}^2x \sum_{\mu=0}^{2}\sum_{a=1}^{3}(\partial_\mu \underline{n}^a)^2}_{=:\ S_0[\underline{n}]} + \underbrace{s}_{\in\mathbb{R}} \cdot H[\underline{n}].$$

So the propagator is:

$$K(n_{\text{Ground}}, -\infty | n_{\text{Ground}}, \infty) = \int \mathcal{D}\underline{n}\ e^{i(S_0[\underline{n}]+sH[\underline{n}])}$$

$$= \sum_{\alpha\in\pi_3(S^2)} \int_{\underline{n}\in\alpha} \mathcal{D}\underline{n}\ e^{isH(\alpha[\underline{n}])}\ e^{iS_0[\underline{n}]} = \sum_{\alpha\in\pi_3(S^2)} e^{isH(\alpha)} \int_{\underline{n}\in\alpha} \mathcal{D}\underline{n}\ e^{iS_0[\underline{n}]}.$$

The topological phase

$$K(n_{\mathrm{G.}}, -\infty | n_{\mathrm{G.}}, \infty) = \sum_{\alpha \in \pi_3(S^2)} e^{isH(\alpha)} \int_{\underline{n} \in \alpha} \mathcal{D}\underline{n}\; e^{iS_0[\underline{n}]}$$

What is special about the topological phase?

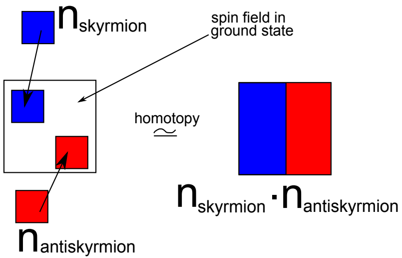
Independent of details of a path!
Depends only on properties like:

- Existence of rotations of a skyrmion
- Existence of skyrmion interchanges
- etc.

Analyze special processes $\rightsquigarrow$ spin and statistics of skyrmions!

Overview

Continuity of pair creation



- $Q[n_{\text{skyrm}} \cdot n_{\text{antiskyrm}}] = Q[n_{\text{skyrm}}] + Q[n_{\text{antiskyrm}}] = 1 + (-1) = 0$
- $Q[n_{\text{ground}}] = 0$

Q is an isomorphism $\Rightarrow n_{\text{skyrm}} \cdot n_{\text{antiskyrm}} \simeq n_{\text{ground}}$
$\rightsquigarrow$ homotopy $h(t, x)$ between them exists
Take the parameter t to be time $\rightsquigarrow$ continuous creation process.

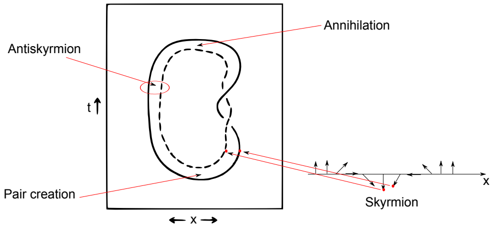
Spin of a skyrmion

Regard the following process:

- n_{ground} at $t = -\infty$.
- Create skyrmion-antiskyrmion pair at some time
- Choose two values of n on the skyrmion.
- Rotate the skyrmion by 2π.
- Annihilate skyrmion-antiskyrmion pair.
- n_{ground} at $t = +\infty$.

What is $H[\underline{n}]$?
Construct the worldlines, then use linking number! ③

Spin of a skyrmion



Without the rotation: $H[\underline{n}_0] = 0$ (or $H[\underline{n}_0] = c \in \mathbb{Z}$).
With rotation: $H[\underline{n}] = 1$ (or $H[\underline{n}] = c + 1$).

$\leadsto$ Rotation of skyrmion produces relative topological phase e^{is}.

Spin of a skyrmion

Recall:
Rotation of a state with angular momentum J by an angle of 2π:

$$U = e^{i2\pi J}.$$

Comparison with $e^{isH[\underline{n}]} = e^{is} \rightsquigarrow$

Result (Spin of skyrmions)

The angular momentum of skyrmions with Hopf term $+sH$ is

$$J = \frac{s}{2\pi}, \quad s \in \mathbb{R}.$$

Statistics of skyrmions

Regard the following process:

- Create two skyrmion-antiskyrmion pairs
- Interchange the two skyrmions
- Annihilate ④

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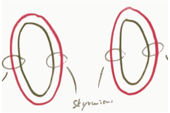
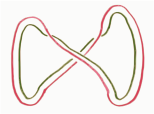
(a)  (b)  (c) 

Worldlines: figures (b), (c) show linking number $\mathrm{Link} = 1 \rightsquigarrow$

Result (Statistics of skyrmions)

The statistical phase of skyrmions with Hopf term $+sH$ is $e^{isH[\underline{n}]} = e^{is}$ for one interchange.

Skyrmions in 3+1 dimensions

For a 3+1 dimensional sigma model:

- For static field configurations $n : \mathbb{R}^3 \to S^2$:
  $\pi_3(S^2) \cong \mathbb{Z}$, so topologically stabilized configurations can exist.
- Including time dependence:

$$\underline{n} : \mathbb{R}_t \times \mathbb{R}^3 \to S^2.$$

  Compactification $\underline{n} : S^4 \to S^2 \rightsquigarrow$ classification in $\pi_4(S^2) \cong \mathbb{Z}_2$.
- Assume there is a topological phase $e^{i\nu}$, with some homomorphism ν defined on $\pi_4(S^2)$:

$$\left(e^{i\nu[\underline{n}]}\right)^2 = e^{i\nu[\underline{n}]+i\nu[\underline{n}]} = e^{i\nu[\underline{n}+\underline{n}]} = e^{i\nu[0]} = e^0 \;\rightsquigarrow\; e^{i\nu[\underline{n}]} = \pm 1,$$

  so such a construction fails to yield anyons in 3+1 dimensions.

Overview

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Summary

- Spin in 2+1 dimensions is not restricted to integer or half-integer values.
- Homotopy theory: continuous deformation of mappings.

- Connection of Hopf term and linking number.
- Hopf term is added to the action of the sigma model.

- Hopf term yields fractional spin and statistics for skyrmions.
- In 3+1 dimensions, the construction does not yield anyons.

A1: Construction of the Hopf term

A1: Construction of the Hopf term

Define the topological current

$$J^\mu := \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\underline{n}^a\varepsilon^{abc}\,\partial_\nu\underline{n}^b\,\partial_\lambda\underline{n}^c, \quad a, b, c \text{ spatial indices.}$$

J^μ is always conserved, independent of the equations of motion (J^μ is *not* a Noether current):
$\partial_\mu J^\mu = 0$ (divergenceless) $\rightsquigarrow$ vector potential A_μ exists:

$$J^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda \quad \text{(curl).}$$

Definition of the Hopf term:

$$H := \int \mathrm{d}t\,\mathrm{d}^2x\; A_\mu J^\mu.$$

A1: Connection of Hopf term and linking number

$\vec{J} = \nabla \times \vec{A}$: Gauge transformation $\vec{A} \mapsto \vec{A} + \nabla\Lambda$ possible.
Coulomb gauge $\nabla \cdot \vec{A} = 0 \rightsquigarrow$ Poisson equation $\Delta\vec{A} = -\nabla \times \vec{J}$.
Solution (cf. electrodynamics):

$$\vec{A} = \frac{1}{4\pi} \int \frac{\nabla_{r'} \times \vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \mathrm{d}^3 r' \overset{\text{int. by parts}}{=} \frac{1}{4\pi} \nabla_r \times \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \mathrm{d}^3 r'.$$

Assumption (current along curve ∂F): $\vec{J}(\vec{r}')\mathrm{d}^3 r' \approx J \mathrm{d}\vec{l}$:

$$\rightsquigarrow \vec{A}(\vec{r}) \approx -\frac{J}{4\pi} \int_{\partial F} \frac{(\vec{r} - \vec{r}') \times \mathrm{d}\vec{l}}{|\vec{r} - \vec{r}'|^3}.$$

Hopf term:

$$H = \int \vec{A} \cdot \vec{J} \mathrm{d}^3 r \approx J \int_{\partial F} \vec{A} \cdot \mathrm{d}\vec{l} \approx -\frac{J^2}{4\pi} \int \int \frac{\left((\vec{r}_1 - \vec{r}_2) \times \mathrm{d}\vec{l}_2 \right) \cdot \mathrm{d}\vec{l}_1}{|\vec{r}_1 - \vec{r}_2|^3}$$

i.e. for $J = 1$ the Gauß integral for the linking number.

A2: Why ν is a homomorphism

A2: Why ν is a homomorphism

General approach to quantum statistics (based on point particles):

- Interchange of particles by transport in position space: Indistinguishability $\rightsquigarrow$ closed path in configuration space. ⑤
- Idea: Describe quantum statistics with closed paths in configuration space.
- $\rightsquigarrow$ classify "inequivalent" (regarding interchange) closed paths by fundamental group (first homotopy group) of the configuration space.

A2: Path integral in non-trivial topology

Configuration space X.

- We know: For a configuration $q \in X$ and action S:
  Propagator from q to q (closed paths):

$$K(q, t_1|q, t_2) = \int_{\tilde{q}(t_1)=\tilde{q}(t_2)=q} \mathcal{D}\tilde{q}\; e^{iS[\tilde{q}]}.$$

- Grouping paths together in equivalence classes:

$$K(q, t_1|q, t_2) = \sum_{\alpha\in\pi_1(X)} \int_{\tilde{q}\in\alpha} \mathcal{D}\tilde{q}\; e^{iS[\tilde{q}]}.$$

  (Remember: $\alpha =$ class of paths which can be deformed into each other.)

A2: Path integral in non-trivial topology

Generalization of the path integral:

- Grouped propagator:

$$K(q, t_1|q, t_2) = \sum_{\alpha \in \pi_1(X)} \int_{\tilde{q} \in \alpha} \mathcal{D}\tilde{q}\, e^{iS[\tilde{q}]}.$$

- Allow a factor $\chi(\alpha) \in \mathbb{C}$:

$$K(q, t_1|q, t_2) = \sum_{\alpha \in \pi_1(X)} \chi(\alpha) \int_{\tilde{q} \in \alpha} \mathcal{D}\tilde{q}\, e^{iS[\tilde{q}]}.$$

Derivation of path integral for one particle: $X = \mathbb{R}^3$. But $\pi_1(\mathbb{R}^3) = \{1\}$, so it is consistent.

A2: Path integral in non-trivial topology

- Conservation of probability $\rightsquigarrow |\chi(\alpha)| = 1$, $\chi(\alpha) = e^{i\nu(\alpha)}$.
- Propagate a particle twice: on $\tilde{q}_1 \in \alpha_1$ and on $\tilde{q}_2 \in \alpha_2$, or concatenate to $\tilde{q}_1 \cdot \tilde{q}_2 \in \alpha_1 \cdot \alpha_2$ and propagate once:
  $\rightsquigarrow \chi(\alpha_1) \cdot \chi(\alpha_2) = \chi(\alpha_1 \cdot \alpha_2)$
  $\rightsquigarrow$ homomorphism/*1D-representation* of $\pi_1(X)$.
- Assign the phase to the states instead of the propagator:
  $\rightsquigarrow$ multivalued states $\Psi_\alpha \approx e^{i\nu(\alpha)}\Psi$, with ordinary propagator $K = \int \mathcal{D}\tilde{q}\; e^{iS[\tilde{q}]}$.

Result (Hopf term)

In the non-linear sigma model: $\pi_1(X) \cong \pi_3(S^2)$. *The Hopf term then yields* $\chi = e^{isH}$ *as a 1D-representation of* $\pi_1(X)$.

A3: Point particles, fermions and bosons

A3: Configuration space of point particles

One particle in $\mathbb{R}^d$. Configuration space of N identical particles?

- Indistinguishability $\rightsquigarrow$ identify permutations:

$$(x_1, \dots x_N) \sim (x_{\sigma(1)}, \dots x_{\sigma(N)}), \quad \sigma \in \mathcal{S}_N.$$

- Allow at most one particle in each place: remove the "diagonal" $\Delta = \{(x_1, \dots x_N) \mid \exists i, j : x_i = x_j\}$.

So the configuration space is

$$X = \left((\mathbb{R}^d)^N \backslash \Delta\right) / \mathcal{S}_N.$$

A3: Configuration space of point particles

Example: $N = 2$, $d = 2$. Then $X = \mathbb{R}^2 \times r_2^2$ ($\mathbb{R}^2$ center-of-mass coordinate, r_2^2 is $\mathbb{R}^2 \backslash \{0\}$ with $\vec{x} \sim -\vec{x}$).
$\rightsquigarrow r_2^2 =$ "cone without the tip", lots of non-homotopic paths (looping $n \in \mathbb{N}$ times around the cone) ⑥
$\rightsquigarrow$ many possibilities. Compare:

Result (3+1 dimensions)

For N particles in $d \geq 3$ space dimensions, the fundamental group of the configuration space is

$$\pi_1(X) = \mathcal{S}_N, \ \textit{the group of permutations.}$$

The only 1D-representations χ of $\mathcal{S}_N$ are:

symmetric (bosons) and antisymmetric (fermions).

A4: The finite energy boundary condition

A4: The finite energy boundary condition

Result (The finite energy boundary condition)

Field configuration (w.l.o.g. only one component) in polar coordinates:

$$n : (\theta, r) \mapsto n(\theta, r).$$

The finite energy condition $E[n] < \infty$ *requires*

$$\lim_{r\to\infty} r||\nabla n|| = \lim_{r\to\infty} r||\frac{\partial n}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial n}{\partial \theta}\vec{e}_\theta|| = 0.$$

We want to show (with some technical assumptions):

$$\Rightarrow n_\infty(\theta) := \lim_{r \to\infty} n(r, \theta) \text{ is constant w.r. to } \theta.$$

Proof: See blackboard. ∎