Topological Excitations II: Hopf Term and Anyons

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Abstract. This is a draft of my talk in the advanced seminar "Quantum field theory of low-dimensional systems" at the University of Stuttgart in 2009. Topics are the theoretical background of anyon physics (including a discussion of statistics in 2+1 and 3+1 dimensions based on the path integral) and the construction of a field theory with anyonic quasiparticles, namely the O(3) non-linear sigma model with Hopf term. I try to give a readable self-contained introduction showing the basic ideas of anyon physics. The organization of the topics is slightly different from the talk.

Contents

1 Introduction to anyon physics (conceptional part)

This is an introduction to the basic concepts of anyon physics, based on the book of Lerda [1] and the book of Khare [2]. Most notions in the following can be thought of as classical; quantum mechanics is included by using the path integral. To make the explanations more graspable I speak of (point) particles here. However, the whole theory will be abused for field theories in the second part of the talk (the difference lies mainly in the choice of the configuration space, but the interpretation is less obvious for fields).

1.1 Spin in different space dimensions

Spin (and angular momentum in general) in 3+1 dimensions (3 space, 1 time) is well-known to obey the following commutator relations:

$$[S_i, S_j] = i\varepsilon_{ijk} S_k.$$

It follows (as is shown in every course on quantum mechanics): The eigenvalues are given by

$$\vec{S}^2 \left|s, m\right\rangle = s(s+1) \left|s, m\right\rangle , \quad s \in \frac{1}{2}\mathbb{N}.$$

Therefor, spin is always integer or half-integer. Furthermore, in 3+1 dimensions, there is the spin-statistics theorem: Particles with s integer are bosons (symmetric states under permutation of identical particles), particles with s half-integer are fermions (antisymmetric states).

Now, in 2+1 dimensions, there is only one axis of rotation (instead of 3 in the 3+1 dimensional case). As a consequence, we have only one operator of angular momentum and no commutator relations. We conclude:

Result 1.1. *In 2+1 dimensions, spin is* not *restricted to integer and half-integer values.*

So, with the spin-statistics theorem in mind, we might suspect that in 2+1 dimensions there are not only bosons and fermions but also particles with other, peculiar, statistics. Indeed, we will construct a theory of such particles, called *anyons*, in the second part of this talk.

1.2 Homotopy groups

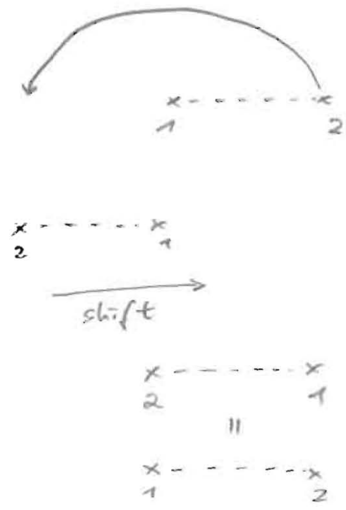
To understand anyons, we need to talk about interchange of particles by transport in position space. Interchange of particles corresponds to closed paths in configuration space (see **drawing** below). In the following, we develop the language to classify closed paths according to the number of particle interchanges they describe (actually, the connection to particle interchanges will become clear only in section 1.4). However, the framework of homotopy theory is much more general than that.

In this section, I will give some fundamental definitions from homotopy theory. I recommend thinking mainly about the relatively intuitive fundamental group (which is also called the first homotopy group) to understand the definitions. The following text is not to be taken as mathematically precise but as a quick intuitive introduction, so it is not formulated in full precision

Why does interchange of particles correspond to closed paths in configuration space?

Closed path in conf. space ≙ evolution of coordinates in time with initial conf. = final conf.

For two particles: identical (→ indistinguishable)



: same conf. as

because of indistinguishability of identical particles.

↳ Particle 2 was not moved on a closed path in ordinary space, but the final configuration is the same as the initial configuration, so the evolution of the system described a closed path in configuration space (≠ physical space).

(eg. we should pay more attention to properties like path-connectedness).

(Important points being: closed paths as continuous mappings $S^1 \to X$, continuous deformation *with one arbitrary point held fixed*, equivalence relation to classify the paths, multiplication of paths and group structure. Discuss the example of $\pi_1(S^1) = \mathbb{Z}$ ("winding number"). Generalization to higher homotopy groups without details.)

In the following, S^n denotes the n dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. First we have to define the notion of a homotopy.

Definition 1.2. *Let X be a topological space.*
A homotopy between two continuous mappings $f_1, f_2 : S^n \to X$ is a continuous mapping $h : S^n \times [0,1] \to X$ with

$$h(x,0) = f_1(x),\ h(x,1) = f_2(x) \quad \forall x \in S^n.$$

Define two mappings f_1, f_2 to be equivalent if there exists a homotopy between them; this is an equivalence relation $\sim$.

For the next step, we choose a point $s_0 \in S^n$ and a point $x_0 \in X$, and look only at mappings with

$$f(s_0) = x_0.$$

We also require all the homotopies to hold this point fixed:

$$h(s_0, t) = x_0 \quad \forall t \in [0,1].$$

Now we can define homotopy groups.

Definition 1.3. *Writing $C(S^n, X)$ for the set of all continuous mappings between S^n and X, with fixed points s_0 and x_0, the n^{th} homotopy group is defined as the set of equivalence classes*

$$\pi_n(X) := C(S^n, X)/\sim .$$

The definition of the multiplication in this group is given in appendix 4.2. $\pi_1(X)$ is also called the fundamental group of X and classifies closed paths/loops in X.

The homotopy group does not depend on the choice of s_0 and x_0, but it is important to fix these points in the beginning.

1.3 Path integral in spaces with non-trivial topology

In this section, we want to generalize the path integral for configuration spaces C with non-trivial topology, that is $\pi_1(C) \neq \{1\}$.

Let $q \in C$ be a configuration. The propagator of a system with action S is known to be

$$K(q, t_1|q, t_2) = \int \mathcal{D}\tilde{q}\ e^{iS[\tilde{q}]}, \quad \tilde{q} : \text{ paths in } C \text{ from } q \text{ to } q,$$

where we only look at propagation from a point q in configuration space to itself (that is propagation on *closed* paths $\tilde{q}$ like in exchanging two particles).

Let the configuration space be C. Taking q as the fixed base point in C, we can make use of the fundamental group to write:

$$K(q,t_1|q,t_2) = \sum_{\alpha\in\pi_1(C)} \int_{\tilde{q}\in\alpha} \mathcal{D}\tilde{q}\; e^{iS[\tilde{q}]}.$$

Remember that α is a whole class of closed paths which can be deformed into each other. The generalization of this expression is:

$$K(q,t_1|q,t_2) = \sum_{\alpha\in\pi_1(C)} \chi(\alpha) \int_{\tilde{q}\in\alpha} \mathcal{D}\tilde{q}\; e^{iS[\tilde{q}]}, \quad \chi(\alpha)\in\mathbb{C}.$$

Why is that possible? There are several reasons I can give:

- The path integral was derived for one particle in $C=\mathbb{R}^3$. In this case $\pi_1(C)=\{1\}$, so the generalized expression reduces to the old expression. So from a mathematical point of view, it is a perfectly valid generalization. (The relevance for physics will become clear soon.)
- At the moment, we can take this generalization as given and derive properties. Later on, we will see how the topological factors arise from the action, so the question reduces to asking if one uses the correct action.

What general properties of $\chi(\alpha)$ can we derive?

- Probability is conserved! This implies

  $$|\chi(\alpha)| = 1,$$

  so $\chi(\alpha)$ is just a phase.
- We can either propagate a particle on a path $\tilde{q}_1\in\alpha_1$ and then again on a path $\tilde{q}_2\in\alpha_2$, or we concatenate both paths to $\tilde{q}_1\cdot\tilde{q}_2\in\alpha_1\cdot\alpha_2$ and propagate the particle only once. The resulting state should be the same. This implies

  $$\chi(\alpha_1)\chi(\alpha_2) = \chi(\alpha_1\cdot\alpha_2).$$

Mathematically speaking:

Result 1.4. *The mapping $\chi:\pi_1(C)\to\mathbb{C}$ is a one-dimensional representation of the fundamental group (one-dimensional here has nothing to do with the dimensionality of spacetime).*

Finally, we state two slight reformulations of the preceding results. To do this, we write $\chi(\alpha)=e^{i\nu(\alpha)}$ and then arrive at

$$K(q,t_1|q,t_2) = \sum_{\alpha\in\pi_1(C)} e^{i\nu(\alpha)} \int_{\tilde{q}\in\alpha} \mathcal{D}\tilde{q}\; e^{iS[\tilde{q}]} = \int_{\text{all } \tilde{q}} \mathcal{D}\tilde{q}\; e^{i(S[\tilde{q}]+\nu(\alpha[\tilde{q}]))}, \tag{1}$$

where the phase has been absorbed in a topological action

$$S_{\text{top}}[\tilde{q}] = S[\tilde{q}] + \nu(\alpha[\tilde{q}]). \tag{2}$$

This formulation will be used in the second part of the talk to construct a field theory with anyons. (Remark: The ν-term can be seen as an interaction in a theory of ordinary bosons. However, solving this problem of interacting bosons is very difficult.)

The second reformulation consists in assigning the phase to the states instead of the propagator. Thereby, states become multivalued: Their phase depends on the path by which they are reached, while the propagator looks like an ordinary propagator now:

$$\Psi_\alpha \approx e^{i\nu(\alpha)}\Psi$$

This suggests calling $\nu(\alpha)$ the *statistics* of a particle since we know that fermions aquire a phase of $e^{i\pi}$ by exchanging them through a transport in configuration space. In the following section, we will see that this definition of statistics contains the ordinary one (usually in 3+1 dimensions, statistics are defined using permutations).

I hope the reader asks why we do not just take $\int \mathcal{D}\tilde{q}\ e^{iS[\tilde{q}]}$ as a phase which we can assign to the state. Or couldn't $\int \mathcal{D}\tilde{q}\ e^{iS[\tilde{q}]}$ be a topological invariant, maybe neutralizing the topological phase $e^{i\nu(\alpha)}$? To answer this question, we have to make an assumption about S. Clearly, we could define $S = -\nu$ or something like that and thus construct pathologies. But in physical theories we take as S a kinetic term, maybe with some additional local interactions. An example is the non-linear sigma model action. In such a realistic case, $S[\tilde{q}]$ does not only depend on the homotopy class of $\tilde{q}$ but also on the details of the path. To give an explicit example: Imagine a system with two particles, one of them fixed at a point q_0. Calculate the propagation of the other particle. It will depend on $\int \mathcal{D}\tilde{q}\ e^{iS[\tilde{q}]}$ and on the topological phase. Now, choose a point $q_0' \neq q_0$ in space and draw a straight line from q_0' to infinity in one direction (example: for particles moving in $\mathbb{R}^N$ this could be $q_0' = 0$ and the line $\{x \in \mathbb{R}^N \mid x_1 > 0, x_2 = \cdots = x_N = 0\}$ – see the **hand drawing**). Let this line be impenetrable and calculate the propagation again. A lot of previously possible paths are now excluded from the path integral, so it can be expected to have a different value now. The topological phase on the other hand is the same as before because we have not created "holes" in space (topologically speaking, our new space is homeomorphic to the old one).

1.4 Statistics and properties of $\pi_1(C)$ in different dimensions

This section aims for showing how in the case of point particles our concept of statistics works and it should clarify the connection with usual treatments of statistics using permutations.

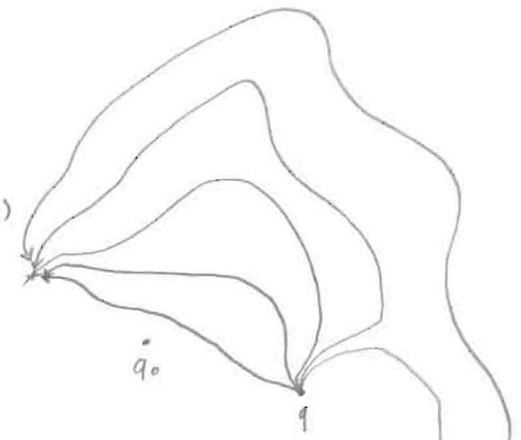
For this section, let M_N^d be the configuration space of N identical point particles in d dimensions. What does this space look like? To make things easier, we take $\mathbb{R}^d$ as the configuration space of *one* particle. For N particles we use N coordinates; so is $M_N^d = (\mathbb{R}^d)^N$? This is not right! Since the particles are indistinguishable, states differing only in a permutation of the coordinates are the same: We identify

$$(x_1, \dots x_N) \sim (x_{\sigma(1)}, \dots x_{\sigma(N)}), \quad \sigma \in \mathcal{S}_N.$$

What's the difference between $\int D\tilde{q}\ e^{iS[\tilde{q}]}$ and $e^{i\nu[\tilde{q}]}$?

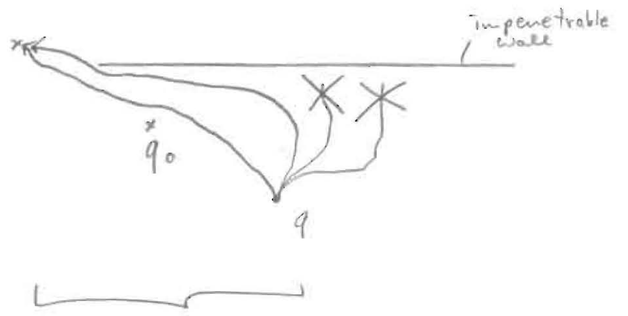
Consider situation 1), then add an impenetrable wall and look at situation 2).
↳ in physical space

1)



vs.

2)



much less paths to integrate over

$$K_1 = \int Dq\ e^{iS[q]}.$$

$$\chi_1 = e^{i\nu[\tilde{q}]}.$$

↳ $K_2 = \int Dq\ e^{iS[q]} \neq K_1.$
(allowed paths.)

$$\chi_2 = \chi_1 .$$

(As both conf. spaces, with wall and without, are homeomorphic (that is, identical from a topological viewpoint).)

⇝ So $e^{i\nu[\tilde{q}]}$ depends only on „interchange or not interchange" while $\int Dq\ e^{iS[\tilde{q}]}$ depends on details of the dynamics.

($\mathcal{S}_N$ is the permutation group permutating N objects). Additionally, let us remove the so-called diagonal

$$\Delta = \{(x_1, \dots x_N) \in (\mathbb{R}^d)^N \mid \exists i, j : x_i = x_j\}.$$

(This means that we do not allow two particles to occupy the same position. This assumption can be justified in two ways: First, if we do not remove the diagonal, the theory can only describe bosons. Second, for anyons and fermions there is an exclusion principle (like the Pauli principle); for bosons we do not need to take out the diagonal.) So we arrive at the following conclusion:

Result 1.5. *The configuration space of N identical point particles is*

$$M_N^d = \left((\mathbb{R}^d)^N \backslash \Delta\right) / \mathcal{S}_N,$$

where $/\mathcal{S}_N$ means the identification as explained above.

At this point, we remember the path integral: It showed us that statistics can be understood by classifying the closed paths using the fundamental group $\pi_1(M_N^d)$ and by looking at its one-dimensional representations. Without proof (which involves some more mathematics) I quote the following result:

Result 1.6. *The fundamental group of the configuration space of N particles in d dimensional space (spacetime: d+1 dimensions) is*

$$\pi_1(M_N^d) = \begin{cases} \mathcal{S}_N & \textit{if } d \geq 3 \\ \mathcal{B}_N & \textit{if } d = 2 \end{cases}.$$

So in 3+1 dimensions (d=3) we have to study only representations of the permutation group $\mathcal{S}_N$. Its only one-dimensional representations are the symmetric and the antisymmetric representation [5], corresponding to bosons and fermions.

On the other hand, in 2+1 dimensions we have the more complicated "braid group" $\mathcal{B}_N$ which has more representations, giving anyonic statistics.

We end the section with a few remarks to make this result more graspable: For $N = 2$, one can write $M_2^d = \mathbb{R}^d \times r_2^d$, where $\mathbb{R}^d$ contains the center of mass coordinate, and r_2^d is $\mathbb{R}^2$ with $r = 0$ taken out and $r \sim -r$ identified. Only the r_2^d part of the configuration space has non-trivial topology. For $d = 3$ one can convince oneself that there are only two sorts of paths. For $d = 2$, r_2^2 is a cone without the tip, which allows for a lot of non-homotopic paths (winding once, twice, thrice, ... around the cone). One should keep this diversity in mind as the essential difference between 2+1 and 3+1 dimensions.

2 Realization of anyons in a non-linear sigma model (applications)

In this section, I will show how a representation of the fundamental group $\pi_1(C)$ arises from a modification of the sigma model Lagrangian. We will find topological excitations as quasiparticles which carry arbitrary spin and obey fractional statistics. This section is based on the article by F. Wilczek and A. Zee [3].

2.1 Repetition (short): Solitons in a non-linear sigma model

At this point, a short repetition of solitons in the O(3) non-linear sigma model is given.

The O(3) non-linear sigma model describes a continuous two-dimensional spin field:

$$n : \mathbb{R}^2 \to S^2, \quad x \mapsto n(x),\ n(x) \text{ a unit vector in } \mathbb{R}^3.$$

The configuration space C consists of these field configurations n.

The energy is given by the (classical) Hamiltonian:

$$E[n] = \int \mathrm{d}^2x \sum_{a=1}^{3} (\nabla n^a)^2,$$

with $\nabla n^a(x) = (\partial_1 n^a(x), \partial_2 n^a(x))$ consisting of the spatial derivatives. Clearly $E[n] \geq 0$. Choose a ground state, e.g. $n(x) = (1,0,0)^T\ \forall x \in \mathbb{R}^2$. This has $\partial_i n^a = 0$ and therefore minimizes the energy. However, the direction of the ground state is arbitrary because a rotation of $(1,0,0)^T$ does not change the energy (O(3) symmetry).

Now we take a look at excitations n of the ground state with finite energy: $E[n] < \infty$. Finite energy requires some sort of rapid decrease of the $\partial_i n^a(x)$ as $|x| \to \infty$. To ensure this, we impose the boundary condition

$$n(x) \to (1,0,0)^T \quad (|x| \to \infty).$$

This allows us to compactify $\mathbb{R}^2$ to S^2 by adding the point "infinity" (one-point compactification) so that n can be seen as a *continuous* mapping $S^2 \to S^2$:

$$\begin{aligned} n :\ & \underbrace{\mathbb{R}^2 \cup \{\infty\} \cong S^2}_{\text{compactified position space}} \to \underbrace{S^2}_{\text{internal space (spin)}} \\ & x \in \mathbb{R}^2 \mapsto n(x),\ \ \infty \mapsto (1,0,0)^T \end{aligned} \tag{3}$$

(The boundary condition is necessary to make $n : S^2 \to S^2$ continuous in the point ∞.) Continuous mappings $S^2 \to S^2$ can be classified in homotopy classes, the elements of $\pi_2(S^2)$. Remember that two field configurations are equivalent by homotopy if they can be continuously deformed into each other. Mathematicians show that

$$\pi_2(S^2) \overset{\phi}{\cong} \mathbb{Z},$$

where the *isomorphism* ϕ is given by the Pontryagin number (topological charge): $\phi : n \mapsto Q[n]$. Time evolution is a continuous deformation of the field. Therefor, a field configuration in the $Q = a$ sector can not evolve into a field configuration with $Q \neq a$. Excitations with $Q \neq 0$ are generally called solitons, excitations with $Q = 1$ are called skyrmions.

Remark 1: E comes from the Legendre transformation of the Lagrangian in the action

$$S[n] = \int \mathrm{d}t\mathrm{d}^2x \underbrace{\sum_{\mu=0}^{2}\sum_{a=1}^{3} (\partial_\mu n^a)^2}_{\text{Lagrangian density}}. \tag{4}$$

Remark 2: In this repetition we have only dealt with static field configurations. Keep in mind the idea of compactification which allows us the use of homotopy groups. This idea will be used in *different* applications in the following. In the next section we will deal with time dependent field configurations, evolving from ground state to ground state with different intermediate processes. The time axis is involved then, so $\pi_3(S^2)$ will be used.

2.2 Introducing the Hopf term

The configuration space of the sigma model consists of field configurations. A closed path in configuration space is a family of field configurations, parameterized by time in a continuous way:

$$n_t : \mathbb{R}^2 \to S^2, \quad t \in S^1 = \mathbb{R} \cup \{\infty\}.$$

(Here, we have compactified the time axis to S^1.) At this point, we have to fix a base point x_0 in configuration space. As a configuration consists of a whole field now, choose x_0 to be the ground state $n_{\text{Ground}}(x) = (1,0,0)^T \; \forall x \in \mathbb{R}^2$. On the time axis S^1, choose[1] $t_0 = \infty = -\infty$. Then every closed path $t \mapsto n_t \in C$ can be interpreted as a mapping

$$n : \mathbb{R}_t \times \mathbb{R}^2 \to S^2, \quad (t,x) \mapsto n(t,x) = n_t(x).$$

By the finite energy requirement,

$$n_t(x) \to (1,0,0)^T \quad (|x| \to \infty).$$

Furthermore, by our choice of the base boint

$$n_t \to (1,0,0)^T = n_{\text{Ground}} \quad (|t| \to \infty).$$

and so $n(t,x) \to (1,0,0)^T$ for $|\underbrace{(t,x)}_{\mathbb{R}^3}| \to \infty$. So by compactifying $\mathbb{R}^3 \cup \{\infty\} = S^3$, n is a continuous mapping

$$n : S^3 \to S^2.$$

Result 2.1. *This shows that every closed path $t \mapsto n_t$ in C (from vacuum to vacuum) can be seen as an element of $\pi_3(S^2)$. With this identification we have an isomorphism such that $\pi_1(C) \cong \pi_3(S^2)$ (see **handwritten** explanation below).*

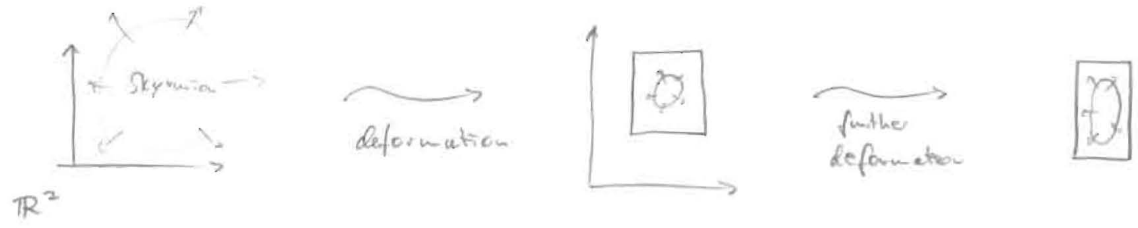
Now we can introduce the *Hopf invariant* (an explicit construction is given in the appendix). At the moment, we need only the following facts:

- Let $n : S^3 \to S^2$ be a vacuum-to-vacuum field evolution. Then $H(n) \in \mathbb{Z}$ and $H(n)$ does not change if n is deformed by a homotopy (H is a homotopic invariant). So H only depends on the class $\alpha \in \pi_3(S^2)$ to which n belongs.
- Furthermore $H : \pi_3(S^2) \to \mathbb{Z}$ is a homomorphism:

$$H(\alpha_1 \cdot \alpha_2) = H(\alpha_1) + H(\alpha_2) \quad \forall \alpha_1, \alpha_2 \in \pi_3(S^2). \tag{5}$$

[1]Here $\infty = -\infty$ because for closed paths the time axis $\mathbb{R}$ is compactified to S^1.

Multiplication in $\pi_2(X)$ as an example:



(squeezed along 1-axis)

Skyrmion  , Antiskyrmion  .

Multiplication

For $\pi_1(C) = \pi_3(S^2)$:

Multiplication in $\pi_3(S^2)$:
Configurations are squeezed along the time axis and then glued one behind the other.
↑ in the sense of temporal order.

This multiplication is the same as concatenating paths in $\pi_1(C)$, so the identification of $\pi_1(C)$ with $\pi_3(S^2)$ is a homomorphism!

!

> For showing that $e^{i\theta H}$ is a representation of $\pi_1(C)$ it is essential that concatenation of paths in $\pi_1(C)$ is equivalent to multiplication in $\pi_3(S^2)$!

Remark: let $u_1 : t \mapsto u_{1,t}$, $u_2 : t \mapsto u_{2,t}$ be paths in C.
Assume there is a homotopy

$$h : \mathbb{R}_t \times [0,1] \longrightarrow h(t,s)$$

with $h(t,0) = u_{1,t} \quad \forall t \in \mathbb{R}_t$, $h(t,1) = u_{2,t} \quad \forall t \in \mathbb{R}_t$.

Now u_i is a mapping $S^3 \to S^2$ and

$$\left.\begin{aligned} h(t,0)(x) &= u_{1,t}(x) = u_1(t,x) \\ h(t,1)(x) &= u_{2,t}(x) = u_2(t,x) \end{aligned}\right\} \quad \forall (t,x) \in S^3 \text{ bzw. } \mathbb{R}^3$$

So u_1 is homotopic to u_2 also if considered as $S^3 \to S^2$, thus the identification is really

$$\pi_1(C) \longrightarrow \pi_3(S^2),$$

a mapping of homotopy classes.

> And $h(t_0, s) = h_0 \ \forall s \in [0,1]$, so $h(t_0, s)(x) = h_0(x) \in S^2$, the base point is respected.

// this relates to result 2.1!

(Three remarks about that:
The definition of a multiplication in $\pi_3(X)$ is more complicated than in $\pi_1(X)$. I refer the reader to appendix 4.2 for more details.
The "multiplication" in $\mathbb{Z}$ as a group is just ordinary addition.
Note that H as a mapping $\pi_3(S^2) \to \mathbb{Z}$ is only well-defined because it depends only on the homotopy class of n.)

2.3 Connection between linking number and Hopf term

Calculating the Hopf number can be quite time-consuming. However, our task is highly simplified by noting the following theorems.

Result 2.2 (Sard's Theorem [7]). *Let n be a mapping $S^3 \to S^2$. Then, almost every point in S^2 will have as its inverse image in S^3 a* collection *of nonintersecting closed curves. ("almost every" in the sense of measure theory)*

The proof of this theorem is quite involved, so we leave it to people who can really prove theorems (mathematicians). Instead, we have a look at how this theorem helps us calculating the Hopf number. A connection to the very intuitive linking number is given by the next theorem:

Result 2.3. *Choose two (arbitrary, but non-critical in the sense of Sard's theorem) values of a field configuration: $n(t_a, x_a)$, $n(t_b, x_b) \in S^2$. Their "wordlines" in $\mathbb{R}_t \times \mathbb{R}^2$ are two collections of closed curves: γ_a and γ_b.*
Then, the following equation holds:

$$H(n) = \mathit{Link}(\gamma_a, \gamma_b),$$

where "Link" is the linking number of the closed curves (see figure 1).

We do not give a proof and prefer to show the applications of this theorem to physics. Two interesting examples are the spin and the statistics of skyrmions, to be discussed in the following sections.

2.4 The topological action

The crucial idea now is: Change the action (4) of our system to be

$$S'[n] = S[n] + \theta H[n], \tag{6}$$

with $\theta \in \mathbb{R}$ an arbitrary parameter! Remember that the sigma model is an effective theory; the existence of a Hopf term in its action should be decided on a microscopic level and will not be discussed in this talk. We just assume it is there. Compare (6) with the equations (1) and (2). It looks like we have found a way to construct the $\chi(\alpha) = e^{i\nu(\alpha)}$-terms in the path integral, realized as $\chi[n] = e^{iH[n]}$ (for n a representative of the equivalence class $\alpha \in \pi_1(C) = \pi_3(S^2)$).

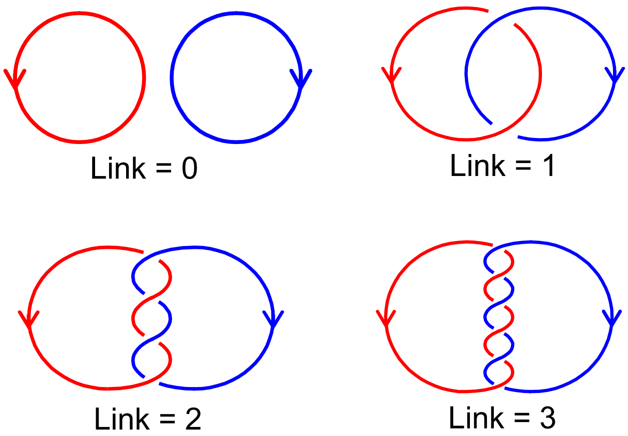


Figure 1: Examples of curves with different linking numbers. The curves have to be interpreted as living in $\mathbb{R}^3$.

We check that this gives us a representation of $\pi_1(C)$. Let n_1, n_2 be paths representating elements of $\pi_1(C)$. As we have seen, they can be identified with elements of $\pi_3(S^2)$. So using result 2.1 we find

$$\chi[n_1] \cdot \chi[n_2] = e^{iH[n_1]} e^{iH[n_2]} = e^{i(H[n_1]+H[n_2])} \overset{(5)}{=} e^{iH[n_1 \cdot n_2]} = \chi[n_1 \cdot n_2].$$

(Ok, I'm tricking you a bit here to make you believe that the Hopf term really describes statistics: As we are working with a field theory now, there is no "moving particles around each other" at this stage. However, a particle interpretation of the soliton excitations will come into the field theory in the next section to convince you.)

2.5 Spin of a Skyrmion

In the following, we will interpret skyrmions as particles. This is based on the following footing: by a continuous deformation a skyrmion can be assumed to live in a very small spacetime region of the field, like a localized nearly-point particle.

At this point, I should explain how creation of skyrmion-antiskyrmion pairs works and why it is a continuous process. A skyrmion has $Q = 1$, an antiskyrmion $Q = -1$, the ground state (vacuum) $Q = 0$. Pair creation should result in a state with $Q = 0 = 1 + (-1)$. This is achieved by taking a skyrmion configuration n_s and an antiskyrmion configuration n_a and create a pair configuration by multiplying them: $n_{\text{pair}} = n_s \cdot n_a$. Here, multiplication means multiplication of elements of $\pi_2(S^2)$ (*static* field configurations composed by placing them side by side, cf.appendix 4.2). So we end up with $Q(n_{\text{pair}}) = Q(n_s \cdot n_a) = Q(n_s) + Q(n_a) = 0$ as intended, and since Q is an isomorphism it follows that n_{pair} and the vacuum belong to the same homotopy class in $\pi_2(S^2)$. Therefore, there is a continuous deformation between n_{pair} and the vacuum, so that pair creation can be realized as a continuous process.

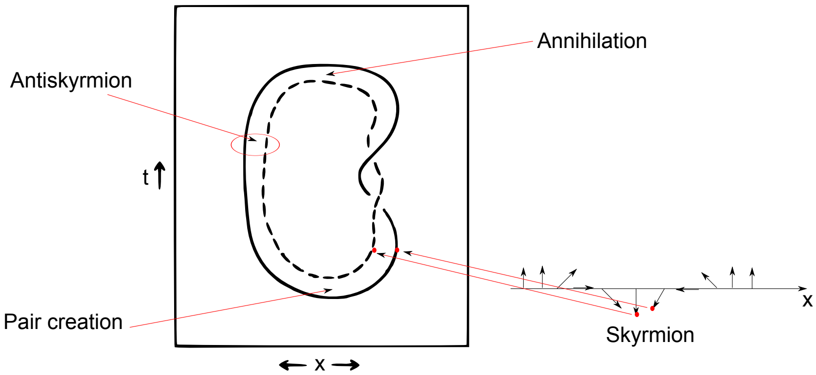


Figure 2: A skyrmion-antiskyrmion pair is created, the skyrmion is rotated by 2π and the pair is annihilated. Shown are the worldlines of two chosen values of the field in the described process. The linking number is 1.

Finally, we arive at the spin of skyrmions. To make it explicit, consider the following time evolution of a field:

- Vacuum at $t = -\infty$
- Create a skyrmion-antiskyrmion pair at some time
- Choose two values of the field which lie on the skyrmion excitation (those points will be used to calculate the linking number)
- Rotate the skyrmion by 2π around itself
- Annihilate the pair
- Vacuum at $t = +\infty$, so we have a closed path in C.

This whole process as a time dependent field configuration is continuous (as explained above). Call it $n \in \pi_3(S^2)$. The Hopf number is given by the linking number. What do the worldlines of the chosen field values look like? It is one loop for each of the two points.

To make this clear, let us first look at the first field value only. By Sard's Theorem, its inverse image could be zero, one, two,... closed curves in spacetime. It can not be zero curves, because we chose values lying on the skyrmion at some time.

Now, if it is exactly one closed curve, at some time we have two space points (see figure 2, the black *or* the dashed line) where the field takes the chosen value. By our choice of the field values one of the space points corresponds to the skyrmion. The one on the other side of the closed curve can then be interpreted as lying on the antiskyrmion.

Now if it were two or more paths, at some time we would have four or more places in space where the field takes the chosen value. But then, four places with the chosen value are too many for just one skyrmion and one antiskyrmion (as long as we assume that they are not fancily deformed).

Finally, look also at the second field value. It gives a second closed loop, which by homotopic deformation of the skyrmion lies next to the other worldline.

The rotation of the skyrmion introduces a twist between the two wordlines. It follows that the worldlines look like[2] in figure 2, which shows that Link $= 1$. So this closed path in configuration space has $\chi[n] = e^{i\theta H[n]} = e^{i\theta}$. Without the rotation we would have found $\chi[n] = 1$, and so the relative phase is $e^{i\theta}$.
Recall the fact that a rotation is also given as $e^{i2\pi J}$ for a state of angular momentum J (since angular momentum generates rotations). The comparison yields:

Result 2.4. *The angular momentum of skyrmions with Hopf term $+\theta H$ is*

$$J = \frac{\theta}{2\pi}, \quad \theta \in \mathbb{R} \textit{ arbitrary.}$$

So it is possible to have J neither integer nor half-integer and we have shown that skyrmions obtain fractional spin by adding the Hopf term. What is left is to study their statistics.

Remark: Solitons with winding number $Q \neq 1$ have $J = \frac{\theta}{2\pi} Q^2$. This can be derived using the canonical formalism. In our derivation, it should come from the internal structure of such solitons: If a field configuration winds $Q \neq 1$ times around S^2 the worldlines can be expected to be more complicated, so that they link Q^2 times.

2.6 Statistics of Skyrmions

The discussion of the statistics of skyrmions works much like the discussion of their spin. The idea now is to regard a process in which two skyrmions are interchanged. Explicitly, the process is:

- Vacuum at $t = -\infty$
- Create two skyrmion-antiskyrmion pairs
- Interchange the two skyrmions
- Annihilate
- Vacuum at $t = \infty$.

The world lines of two field values are shown in figure 3. (This time two identical skyrmions exist, so as long as one doesn't do any interchange, each field value has two loops as its inverse image – black lines for the first value, dashed lines for the other value.) The figure shows that the linking number and therefor the Hopf invariant is 1. (Without the interchange, we are in case (a) of figure 3 with Link $= 0$.) It follows that the field picks up a phase $e^{i\theta}$ in this process *because of the interchange*. This can be compared to bosons with a phase $e^{i0} = 1$ (like in the non-linear sigma model without Hopf term) or fermions with $e^{i\pi} = -1$.

[2]Actually, the worldlines could be already twisted even without the skyrmion being rotated. But then, rotating the skyrmion would introduce an *additional* twist, resulting in the same relative phase.

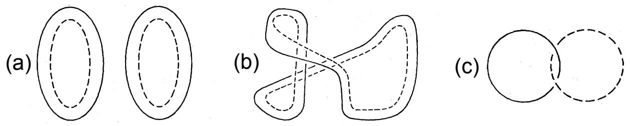


Figure 3: (a) Creation and annihilation of two skyrmion-antiskyrmion pairs. (b) Same, but with interchange of the two skyrmions. (c) The curves in (b) after homotopic deformation.

Result 2.5. *The statistical phase of skyrmions (with Hopf term) is* $e^{i\theta}$ *for one interchange, with an arbitrary parameter* $\theta \in \mathbb{R}$*. This means that in general skyrmions are neither bosonic nor fermionic.*

Remark: It is $\pi_4(S^2) = \mathbb{Z}_2$, so $(\chi[n])^2 = \chi[n \cdot n] = \chi[e] = 1$ (e: neutral element) and therefor $\chi[n] = \pm 1$ and thus we have only bosons or fermions in the model with 3+1 spacetime dimensions.

3 Summary

Let us end with a short overview of the discussed topics.

- Spin in 2+1 dimensions is not restricted to integer and half-integer values.
- Homotopy is introduced. Especially, paths in configuration space are equivalent if they can be continuously deformed into each other.
- The path integral is generalized for configuration spaces with non-trivial homotopy groups.
- Statistics are described by one-dimensional representations of $\pi_1(C)$, C being the configuration space. 2+1 dimensions allow more kinds of statistics than just bosonic and fermionic.
- The Hopf term is added to the action of the non-linear sigma model.
- The connection between Hopf term and linking number is explained.
- The Hopf term is shown to yield fractional spin and statistics for the skyrmion.

4 Appendix

4.1 Construction of the Hopf term

If there is still enough time, we can construct the Hopf term explicitly. Define the topological current as

$$J^\mu := \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda} n^a \varepsilon^{abc} \partial_\nu n^b \partial_\lambda n^c, \quad \mu = 0,1,2, \ a = 1,2,3.$$

This current is always conserved, $\partial_\mu J^\mu = 0$, independent of the equations of motion (it is *not* a Noether current). The corresponding charge is the winding number/Pontryagin number Q used in classifying the static field configurations. As the divergence of J^μ vanishes, there is a vector potential A_μ which has J^μ as its curl:

$$J^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda.$$

Then the Hopf term is defined as

$$H[n] = -\frac{1}{2\pi}\int \mathrm{d}t\mathrm{d}^2x A_\mu[n]J^\mu[n].$$

4.2 Definition of the multiplication in higher homotopy groups

Here I'll give a short explanation of how one defines the multiplication on $\pi_n(X)$ (in case you are not comfortable with my previous drawings on the black board). More details can be found in books on algebraic topology. Paths ($n = 1$) can easily be multiplicated by concatenation. For $n > 1$ we have to clarify some other definitions first before we define the multiplication.

Let (X, x_0) be a topological space with a point $x_0 \in X$ chosen. Let $I^n := [0,1] \times \cdots \times [0,1]$ be the unit n-cube with boundary $\partial I^n = \{(s_1, \ldots s_n) \in I^n | \exists i : s_i = 0 \text{ or } s_i = 1\}$. A closed n-loop is a continuous map $\alpha : I^n \to X$ with $\alpha(\partial I^n) = \{x_0\}$. Let $I^n/\partial I^n$ denote the n-cube with ∂I^n shrunk to a point, then we see that $I^n/\partial I^n \cong S^n$. So effectively $\alpha : S^n \to X$ and the new definition reproduces the old one.

The advantage over the old definition is that we can now define the product of two n-loops α and β:

$$\alpha \cdot \beta(s_1, \ldots s_n) := \begin{cases} \alpha(2s_1, s_2, \ldots s_n) & 0 \leq s_1 \leq 1/2 \\ \beta(2s_1 - 1, s_2, \ldots s_n) & 1/2 < s_1 \leq 1 \end{cases}$$

As all boundary points map to x_0 this is a well-defined continuous mapping $S^n \to X$. The homotopy classes of such loops with this multiplication then fulfil the group axioms (without proof). They form the n^{th} homotopy group. For $I^n = I^2$ one has an easy picture of the multiplication: Both field configurations are squeezed along the s_1 direction (which as as homeomorphism does not really change anything) and then glued together side by side.

References

[1] A. Lerda: *Anyons – Quantum Mechanics of Particles with Fractional Statistics*, Lecture Notes in Physics m14. Springer, Berlin Heidelberg 1992.

[2] A. Khare: *Fractional Statistics and Quantum Theory*. World Scientific, Singapore 1997.

[3] F. Wilczek, A. Zee: Phys. Rev. Lett. 51, 2250 (1983).

[4] Y.S. Wu, A. Zee: Phys. Lett. 147B, 325 (1984).

[5] `http://en.wikipedia.org/wiki/Representation_theory_of_the_symmetric_group`, 13th May 2009.

[6] E. Fradkin: *Field theories of condensed matter systems*, Frontiers in physics no. 82. Addison-Wesley, Redwood 1991.

[7] John Baez: `http://math.ucr.edu/home/baez/braids/node3.html`