A little fact about the theory of BEC, related to the

Gross-Pitaevskii energy

GS energy of a BEC? Let's start by writing down a model:

$$H_N = \int dx\, a_x^\dagger (-\Delta_x + V_{ext}(x)) a_x$$
$$+ \frac{1}{2} \int dx\, dy\, \underbrace{N^2 V(N(x-y))}_{\text{short-range, models dilute limit in trap.}} a_x^\dagger a_y^\dagger a_y a_x .$$

$N \overset{\text{going to be}}{=} \#$ particles

Easiest ansatz: perfect condensate

$\longrightarrow$ coherent state:

trial state $\Psi = W(\sqrt{N}\varphi)\,\Omega$ — W: Weyl op. — φ: condensate wave fct.

$$W^*(\sqrt{N}\varphi)\, a_x\, W(\sqrt{N}\varphi) = a_x + \sqrt{N}\varphi(x).$$

Calculate energy:

$$\langle \Psi, H_N \Psi\rangle = \langle \Omega, \int dx\, (a_x^\dagger + \sqrt{N}\overline{\varphi(x)})(-\Delta_x + V_{ext}(x))(a_x + \sqrt{N}\varphi(x))$$
$$+ \frac{1}{2}\int dx\, dy\, N^2 V(N(x-y)) (a_x^\dagger + \sqrt{N}\overline{\varphi(x)})(a_y^\dagger + \sqrt{N}\overline{\varphi(y)})$$
$$\times (a_y + \sqrt{N}\varphi(y))(a_x + \sqrt{N}\varphi(x)).$$

0 0 only these survive

$$= N\Big[\int dx\, |\nabla_x \varphi(x)|^2 + |\varphi(x)|^2 V_{ext}(x)$$
$$+ \frac{1}{2}\int dx\, dy\, \underbrace{N^3 V(N(x-y))}_{\to\, b\,\delta(x-y),\quad b = \int V dx.} |\varphi(x)|^2 |\varphi(y)|^2\Big]$$

②

$$\longrightarrow N \int dx\, |\nabla \varphi|^2 + |\varphi|^2 V_{ext} + \frac{b}{2} |\varphi|^4 .$$

Physicists: obviously non-sense:
low-energy → profile of V should not play a role,
particles see each other as spheres of a certain radius which is the s-wave zero-energy scattering length.

↳ situation was saved when they noticed that $\frac{b}{2}$ = ess. lowest Born approx. to scatt. length.

$$\rightsquigarrow \mathcal{E}_{GP}(\varphi) = \int dx\, |\nabla \varphi|^2 + |\varphi|^2 V_{ext} + 4\pi a_0 |\varphi|^4 . \quad (*)$$

↑ scatt. length.

A bit unsatisfactory
⟶ find trial state that gives the fct. (*) directly by calculation.

As a side result on dynamics we noticed: // Also known to others before ...

For $\Psi = W(\sqrt{N}\varphi)\, T\, \Omega$:

$$\langle \Psi, H_N \Psi \rangle \implies N\, \mathcal{E}_{GP}(\varphi) \qquad (**)$$

$$T = e^{\frac{1}{2}\int dx\, dy \left[k(x;y)\, a_x^* a_y^* - \overline{k(x;y)}\, a_x a_y \right]}$$ (3)

(with a k that we be write down if one has a few minutes more to introduce concepts, but let me just say:)

T is Bog. trafo:

$$T^* a_x^* T = \int dy\, a_y^* \cosh(k)(y;x) + \int dy\, a_y \sinh(k)(y;x)$$

$$\simeq a_y^* + \int dy\, a_y\, k(y;x).$$

this is nice b/c. can plug this in and do the calculation.

$\Longrightarrow$ get $(**)$.

If you want to know more details and what this has to with the dynamics you can buy our book (or ask me) :).

(Springer Briefs in MathPhys 7)