Seminar talk: The Fredholm Alternative

Advanced Mathematical Physics

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This note follows closely Reed and Simon [1]. We let $\mathcal{H}$ denote a seperable Hilbert space, and $\mathcal{L}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$.

Definition 1. Let $U \subset \mathbb{C}$ be open. An operator-valued function $L : U \to \mathcal{L}(X)$ is called *analytic* if the complex-valued function given by

$$z \mapsto f(L(z)x), \qquad \text{for all } z \in U,$$

is analytic for all $x \in X$ and for all $f \in X^*$.

Lemma 1 (Neumann series)**.** *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $\|T\| < 1$ then $I - T$ is bijective and the inverse is bounded. Moreover*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Theorem 2 (Analytic Fredholm theorem)**.** *Let U be an open connected subset of $\mathbb{C}$. Let $L : U \to \mathcal{L}(\mathcal{H})$ be an analytic operator-valued function such that $L(z)$ is compact for each $z \in U$. Then either $(I - L(z))^{-1}$ exists for no $z \in U$ or for all $z \in U \backslash S$, where S is a discrete subset of U. In the latter case $(I - L(z))^{-1}$ is meromorphic in U, analytic in $U \backslash S$, the residues at the poles are finite rank operators, and if $z \in S$ then $L(z)\psi = \psi$ has a nonzero solution in $\mathcal{H}$.*

Proof. Let $z_0 \in U$ and choose $r > 0$ such that $B_r(z_0) \subset U$ and such that $\|L(z) - L(z_0)\| < 1/2$ whenever $|z - z_0| < r$. This is possible since U is open and since L is analytic and hence continuous. Since the finite rank operators are dense in the compact operators, we can pick an operator F of finite rank such that $\|L(z_0) - F\| < 1/2$. Then $\|L(z) - F\| < 1$ for all $z \in B_r(z_0)$. It follows by lemma 1 that $I - L(z) + F$ is invertible and

$$(I - L(z) + F)^{-1} = \sum_{n=0}^{\infty} (L(z) - F)^n.$$

Since $z \mapsto L(z)$ is analytic by assumption, $z \mapsto (I - L(z) + F)^{-1}$ is clearly analytic as well.

As F has finite rank we can find linearly independent vectors $\psi_1, \psi_2, \dots, \psi_N \in \mathcal{H}$ such that $F(\varphi) = \sum_{i=1}^N \alpha_i(\varphi)\psi_i$. for all $\varphi \in \mathcal{H}$. For each $1 \leq i \leq N$, α_i is a bounded linear functional on $\mathcal{H}$, so by Riesz we can find a vector $\xi_i \in \mathcal{H}$ such that $\alpha_i(\,\cdot\,) = \langle \xi_i\,, \cdot\,\rangle$. Then

$$F(\varphi) = \sum_{i=1}^{N} \langle \xi_i\,, \varphi \rangle\, \psi_i, \qquad \text{for all } \varphi \in \mathcal{H}.$$

Define $\xi_i(z) := ((I - L(z) + F)^{-1})^* \xi_i \in \mathcal{H}$ and define $G(z) \in \mathcal{L}(\mathcal{H})$ by

$$G(z)\varphi = F(I - L(z) + F)^{-1}\varphi = \sum_{i=1}^{N} \left\langle \xi_i\,, (I - L(z) + F)^{-1}\varphi \right\rangle \psi_i = \sum_{i=1}^{N} \langle \xi_i(z)\,, \varphi \rangle\, \psi_i.$$

for all $\varphi \in \mathcal{H}$. We can now write

$$I - L(z) = (I - G(z))(I - L(z) + F), \tag{1}$$

and it is clear that $I - L(z)$ is invertible if and only if $I - G(z)$ is invertible, and that $L(z)\psi = \psi$ has a nonzero solution if and only if $G(z)\varphi = \varphi$ has a nonzero solution.

The set $\{\psi_i : 1 \leq i \leq N\}$ can be expanded to an orthogonal basis for $\mathcal{H}$, and we can write any $\varphi \in \mathcal{H}$ as $\varphi = \sum_{i=1}^{\infty} \beta_i \psi_i$. Suppose φ is a solution to $G(z)\varphi = \varphi$. Then

$$\sum_{i=1}^{N} \langle \xi_i(z) , \varphi \rangle \psi_i = \sum_{i=1}^{\infty} \beta_i \psi_i,$$

hence we must have $\beta_i = 0$ for $i > N$. Moreover

$$\sum_{i,j=1}^{N} \beta_j \langle \xi_i(z) , \psi_j \rangle \psi_i = \sum_{i=1}^{N} \left\langle \xi_i(z) , \sum_{j=1}^{N} \beta_j \psi_j \right\rangle \psi_i = \sum_{i=1}^{N} \beta_i \psi_i,$$

so by linear independence we get that β_i satisfies the linear equation

$$\beta_i = \sum_{j=1}^{N} \beta_j \langle \xi_i(z) , \psi_j \rangle , \tag{2}$$

for each $1 \leq i \leq N$. Conversely if equation (2) is satisfied, then $\varphi = \sum_{i=1}^{N} \beta_i \psi_i$ is a solution to $G(z)\varphi = \varphi$. Now equation (2) has a non-trivial solution if and only if

$$d(z) := \det[\delta_{ij} - \langle \xi_i(z) , \psi_j \rangle]_{i,j} = 0.$$

Recall that $(I - L(z) + F)^{-1}$ is analytic in $B_r(z_0)$, ξ_i is a vector in $\mathcal{H}$ and $\psi \mapsto \langle \xi_i , \psi \rangle$ is a linear functional on $\mathcal{H}$. Thus $z \mapsto \langle \xi_i(z) , \psi_j \rangle$ is analytic in $B_r(z_0)$, and then so is $d(z)$. It is a result from complex analysis (theorem 6.3 in [2]) that either the set of zeroes of $d(z)$ is discrete, or $d(z)$ is constant zero, i.e., if we let $S_r := \{z \in B_r(z_0) : d(z) = 0\}$, then S_r is either a discrete set or $S_r = B_r(z_0)$.

Now $I - G(z)$ is invertible if and only if given $\eta \in \mathcal{H}$ we can find $\varphi \in \mathcal{H}$ such that $(I - G(z))\varphi = \eta$. We may without loss of generality write $\varphi = \eta + \sum_{i=1}^{\infty} \beta_i \psi_i$. Clearly for φ to be a solution to $(I - G(z))\varphi = \eta$ we need to have $\beta_i = 0$ for $i > N$. So consider the vector $\varphi = \eta + \sum_{i=1}^{N} \beta_i \psi_i$ in $\mathcal{H}$.

$$(I - G(z))\varphi = \eta + \sum_{i=1}^{N} \beta_i \psi_i - \sum_{i=1}^{N} \langle \xi_i(z) , \eta \rangle \psi_i - \sum_{i,j=1}^{N} \beta_j \langle \xi_i(z) , \psi_j \rangle \psi_i.$$

Then φ is a solution to $(I - G(z))\varphi = \eta$ if and only if for each $1 \leq i \leq N$

$$\beta_i - \langle \xi_i(z) , \eta \rangle - \sum_{j=1}^{N} \beta_j \langle \xi_i(z) , \psi_j \rangle = 0.$$

Define the $N \times N$-matrix

$$M(z) := (I - [\langle \xi_i(z) , \psi_j \rangle]_{i,j}),$$

and note that $M(z)$ is an analytic operator on the finite dimensional vector space $\mathbb{C}^N$. Notice also that $d(z)$, that we defined previously, is the determinant of $M(z)$. By letting

$\alpha(z,\eta) = (\langle \xi_1(z)\,,\eta\rangle\,,\dots,\langle \xi_N(z)\,,\eta\rangle) \in \mathbb{C}^N$ and $\beta = (\beta_1,\dots,\beta_N) \in \mathbb{C}^N$ we can write the above in the slightly more illuminating form

$$M(z)\beta = \alpha(z,\eta).$$

We see that this has a non-trivial solution if and only if $d(z) \neq 0$. Hence $I - G(z)$, and thereby $I - L(z)$, is invertible if and only if $z \notin S_r$. Since $I - L(z)$ is analytic in $B_r(z_0)$, $(I - L(z))^{-1}$ is analytic in $B_r(z_0)\backslash S_r$.

In the case where S_r is a discrete subset of $B_r(z_0)$ and $z \notin S_r$ we have

$$\beta = M(z)^{-1}\alpha(z,\eta).$$

The inverse of $M(z)$ is given by $A(z)/d(z)$, where $A(z) = [a_{ij}]_{i,j=1}^N$ is the adjugate matrix, i.e., the transpose of the matrix of cofactors. The entries of $A(z)$, a_{ij} for $1 \le i,j \le N$, are also polynomials in the entries of $M(z)$. Hence $M(z)^{-1}$ is analytic in $B_r(z_0)\backslash S_r$. From the vector β we obtain $(I - G(z))^{-1}\eta = \eta + \sum_{i=1}^N \beta_i\psi_i$. This operator is analytic in $B_r(z_0)\backslash S_r$ and its singularities are the points of S_r. Define the operator $K(z) := (I - G(z))^{-1} - I$.

$$K(z)\eta = \sum_{i=1}^N \beta_i\psi_i = \sum_{i=1}^N \left(\frac{1}{d(z)} \sum_{j=1}^N a_{ij}(z)\,\langle \xi_j(z)\,,\eta\rangle \right) \psi_i.$$

It is clear that the image of $K(z)$ is a supspace of the vector space spanned by $\psi_1,\dots,\psi_N$. Hence $K(z)$ has finite rank. Recall that $d(z)$ is the determinant of $M(z)$, and recall that $d(z)$ is a non-trivial analytic function and that S_r is its zero set. In particular for $z_c \in S_r$, there exist an integer $n_c \ge 1$ and an analytic function $g : B_{r_c}(z_c) \to \mathbb{C}$ with $g(0) \neq 0$ such that

$$d(z) = (z - z_c)^{n_c} g(z), \text{ for } z \in B_{r_c}(z_c).$$

Hence the following limit

$$\lim_{z\to z_c} (z - z_c)^{n_c} (I - G(z))^{-1} = \lim_{z\to z_0} (z - z_c)^{n_0} K(z)$$

exists and is a non-zero operator. Also z_c is a pole for $(I - G(z))^{-1}$ which has order smaller than n_c. We conclude that $(I - G(z))^{-1}$ is meromorphic in $B_r(z_0)$. From equation (1) we conclude, that the same is true for L. Moreover it is clear that the above limit is of finite rank, and the limit $\lim\limits_{z\to z_j} (z - z_j)^{a'_j} (I - L(z))^{-1}$ is then of finite rank as well, as the set of finite rank operators form an ideal.

To finish the proof we need to extend to U the above discussion. We have shown that for all $z_0 \in U$, there exists a ball $B_r(z_0) \subset U$ such that

- Either $(I - L(z))^{-1}$ is meromorphic on $B_r(z_0)$,
- Or $(I - L(z))^{-1}$ does not exist at all on $B_r(z_0)$.

The set of points satisfying the first proposition (resp. the second) is open by construction. Both are closed because if we consider a sequence $z_n \to z_0 \in U$ with, then z_0 is in one of these two sets, and the z_n's are in the corresponding ball $B_r(z_0)$ for n large enough.

As U is connected, this implies that one of these two sets coincides with U and the other one is empty. □

Corollary 3 (the Fredholm alternative). *If A is a compact operator on $\mathcal{H}$, then either $(I - A)^{-1}$ exists or $A\psi = \psi$ has a solution.*

Proof. Define $L(z) : \mathbb{C} \to \mathcal{L}(\mathcal{H})$ by $L(z) = zA$. Then $L(z)$ is an analytic operator-valued function such that $L(z)$ is compact for each $z \in \mathbb{C}$. Hence the above theorem applies. In particular we get the statement of the Fredholm alternative at $z = 1$. $\square$

The following theorem by Riesz and Schauder may also be proved using the framework we have developed in this note.

Theorem 4 (Riesz-Schauder theorem)**.** *Let A be a compact operator on $\mathcal{H}$. Then $\sigma(A)$ is a discrete set with no limit points except perhaps zero. Moreover any non-zero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity.*

Proof. Define $L(z) : \mathbb{C} \to \mathcal{L}(\mathcal{H})$ by $L(z) = zA$. Then $L(z)$ is an analytic operator-valued function such that $L(z)$ is compact for each $z \in \mathbb{C}$. The set $C := \{z \in \mathbb{C} : zA\psi = \psi$ has a solution $\psi \neq 0\}$ is discrete by the proof of theorem 2 since it does not contain $z = 0$. It is even so, that all points are isolated. Moreover if $1/\lambda$ is not in C then $(I - A/\lambda)$ is invertible, and then so is $(\lambda - A)$. To see this, note that

$$(\lambda - A)^{-1} = \frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1}.$$

Hence $\lambda \notin \sigma(A)$. By contraposition $\lambda \in \sigma(A)$ implies $1/\lambda \in C$. As C is discrete we conclude that $\sigma(A)$ is discrete as well. Further if $z \in \mathbb{C}$ was a non-zero limit point of $\sigma(A)$, then $1/z$ would be a limitpoint of C. As all points in C are isolated, only $z = 0$ can be a limitpoint of $\sigma(A)$.

Suppose $\lambda \in \sigma(A)$ is an eigenvalue of A, and suppose for contradiction that the corresponding eigenspace where infinite dimensional. Let $\{\psi_i\}_{i=1}^{\infty}$ be the set of linearly independent eigenvectors, and let $S = \mathrm{span}\{\psi_i : i \geq 1\} \cap (H)_1$ be all vectors in the span of the eigenvectors of length less than or equal to one. Since A is compact and S is bounded the closure of the image of S under A is compact. But

$$\overline{A(S)} = \lambda\overline{S},$$

and $\overline{S}$ is the closed unitball of the infinite dimensional Banach space $\overline{\mathrm{span}\{\psi_i : i \geq 1\}}$. Hence every non-zero eigenvalue of A has finite multiplicity. $\square$

References

[1] Michael Reed and Barry Simon. *Functinal analysis, Volume I.* Academic Press, New York, 1980.

[2] Christian Berg. *Complex Analysis.* Department of Mathematical Sciences, University of Copenhagen, 2014.