LEMMA 5.5: If Ω is an isometry, then ran Ω is closed and

$$\Omega^* = \begin{cases} \Omega^{-1} & \text{auf ran } \Omega \\ 0 & \text{auf } (\text{ran } \Omega)^\perp . \end{cases}$$

(An isometry (also called "partial isometry") need not be surjective.)

PROOF: Isometries are always injective:

$\Omega\psi_1 = \Omega\psi_2 \Rightarrow 0 = \|\Omega(\psi_1 - \psi_2)\| = \|\psi_1 - \psi_2\| = 0 \Rightarrow \psi_1 = \psi_2$.

So $\Omega : \mathcal{H} \longrightarrow \text{ran } \Omega$ is invertible.

By polarization

$$\langle \varphi, \psi \rangle = \frac{1}{4}\left(\|\varphi + \psi\|^2 - \|\varphi - \psi\|^2 + i\|\varphi + i\psi\|^2 - i\|\varphi - i\psi\|^2 \right) \qquad \forall \varphi, \psi \in \mathcal{H}$$

also scalar products are invariant:

$$\langle \psi, \varphi \rangle = \langle \Omega\psi, \Omega\varphi \rangle = \langle \psi, \Omega^*\Omega\varphi \rangle \quad \forall \varphi, \psi \in \mathcal{H}$$

$\Rightarrow \Omega^*$ is a left-inverse to Ω.

Also a right inverse?

For $\varphi \in \operatorname{ran} \Omega$: $\exists \psi \in \mathcal{H}: \varphi = \Omega \psi$, so:

$$\Omega \Omega^* \varphi = \Omega \underbrace{\Omega^* \Omega}_{=1 \text{ as above}} \psi = \Omega \psi = \varphi.$$

For $\varphi \in (\operatorname{ran} \Omega)^\perp$: $\langle \Omega \psi, \varphi \rangle = 0 \quad \forall \psi \in \mathcal{H}$

$$\Rightarrow \Omega^* \varphi = 0.$$

$\blacksquare$

RMK: Let $H = H^*$, $H_0 = -\Delta/2$ on $L^2(\mathbb{R}^n)$ and assume $\Omega_\pm$ exists.

Let $\mathcal{H}_\pm := \operatorname{ran} \Omega_\pm$, $\mathcal{H}_B := \overline{\operatorname{span}\{\text{eigenvectors of } H\}}$.

"scattering states" „<u>bound states</u>"

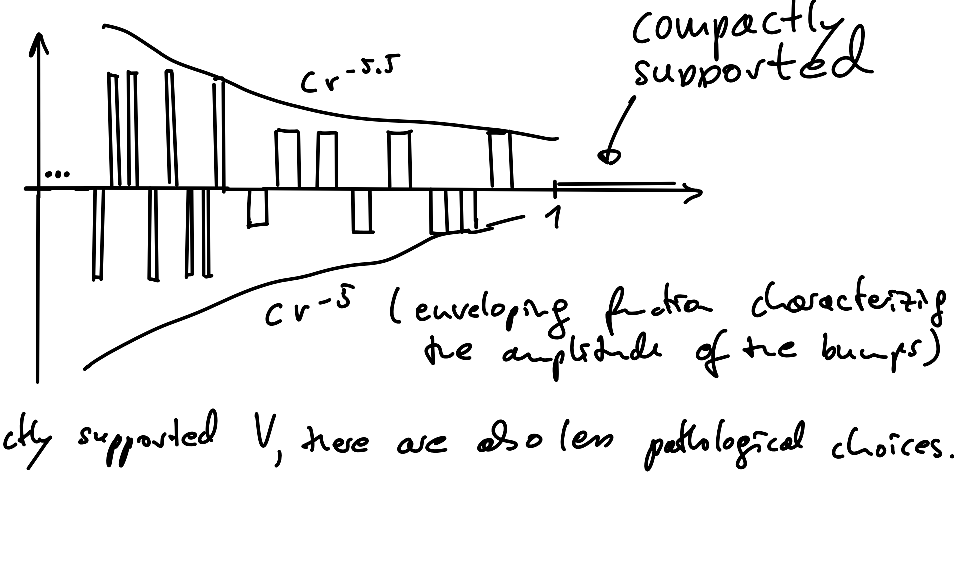
We know already: $\mathcal{H}_\pm$ is closed and $\mathcal{H}_\pm \subset \mathcal{H}_B^\perp$.

Under the assumptions of Prp. 5.3 we have <u>asymptotic completeness (a.c.)</u>:
$\mathcal{H}_+ = \mathcal{H}_B^\perp = \mathcal{H}_-$, i.e. all $\varphi \in \mathcal{H}_B^\perp$ are asymptot. free.

RMK: Asymptotic completeness is a natural expectation: it says that all states are either scattering states or bound states and nothing else.
But a.c. is generally very hard to proof!
And there are counter examples!

EXAMPLE: (not a.c. system — see Reed & Simon)

$V(x) = W(|x|)$



Among non-compactly supported V, there are also less pathological choices.

In physics, scattering is described by probabilities to scatter from a specified ingoing asymptotic state to a specified outgoing asymptotic state. Described by matrix elements of the S-matrix. The following def. makes this rigorous.

DEF: Let $\alpha, \beta \in \mathcal{H}$. Consider ingoing asymptotics $e^{-iH_0 t}\alpha$.
What's the probability to observe outgoing asymptotics $e^{-iH_0 t}\beta$?

$$\| e^{-iHt}\varphi - e^{-iH_0 t}\alpha \| \longrightarrow 0 \quad (t \to -\infty) \quad \Rightarrow \quad \varphi = \Omega_- \alpha .$$

negative infinity

positive infinity

Probability to find at $t \to +\infty$ the state $e^{-iH_0 t}\beta$:

axiom of QM

$$P = \lim_{t \to +\infty} |\langle e^{-iH_0 t}\beta, e^{-iHt}\varphi \rangle|^2 = \lim_{t \to +\infty} |\langle e^{iHt} e^{-iH_0 t}\beta, \Omega_- \alpha \rangle|^2$$

$$= |\langle \Omega_+ \beta, \Omega_- \alpha \rangle|^2 = |\langle \beta, \underbrace{(\Omega_+^* \Omega_-)}_{=:\, S} \alpha \rangle|^2$$

$=: S$, the <u>scattering operator, S-matrix</u>.

If we have a.c.: $S = \Omega_+^{-1}\Omega_-$, unitary.

PROOF OF A.C.: In Teschl you can find the proof by Enss of A.C. for operators $H = -\Delta/2 + V$.

The proof of A.C for N-body systems was given much later by Sigal and Soffer (by abstract method) and Graf in 1990 by a more instructive method.

Here we prove only the <u>simplest case</u> of asympt. compl., that is, when the potential is actually so <u>weak</u> that there are <u>no bound states</u> at all: $\longrightarrow$ Thm. 5.6.

THM. 5.6: Let $H_0 = -\Delta/2$ in $L^2(\mathbb{R}^3)$, $\tilde{V} \in L^1 \cap L^\infty(\mathbb{R}^3)$ real-valued, $H = H_0 + \lambda \tilde{V}$, $\lambda \in \mathbb{R}$. Then the wave operators $\Omega_\pm$ exist. Furthermore, for $|\lambda|$ small enough: $\operatorname{ran} \Omega_\pm = L^2(\mathbb{R}^3)$.

i.e. the whole Hilbert space consists of only scattering states.

RMK: Implies: in 3D there are <u>no</u> bound states if potential is weak.

This is <u>false</u> in 1D, 2D: there are bound states for arbitrarily small attractive potential!

PROOF OF 5.6: Existence of $\Omega_\pm$: see Prp. 5.3. $\square$

<u>Proof that $\operatorname{ran} \Omega_+ = L^2(\mathbb{R}^3)$:</u>

We show: $\lim_{t\to\infty} \underbrace{e^{iH_0t} e^{-iHt}}_{H_0 \leftrightarrow H} \varphi$ exists $\forall \varphi \in L^2(\mathbb{R}^3)$.

This is a general principle: a.c. is equivalent to existence of wave op.' with H_0 and H exchanged.

Then $\exists \gamma \in L^2(\mathbb{R}^3)$: $e^{iH_0t} e^{-iHt}\varphi \to \gamma \Leftrightarrow \underbrace{\lim_{t\to\infty} e^{iHt} e^{-iH_0t}\gamma}_{=\Omega_+\gamma} = \varphi$

$\Rightarrow \operatorname{ran} \Omega_+ = L^2(\mathbb{R}^3)$. $\square$

By Lm. 4.1, for existence of the s-lim it is sufficient to consider vectors φ from a dense subset, since $\| e^{iHt} e^{-iH_0t} \| = 1$.

Suffices to consider $\varphi \in \mathcal{S}(\mathbb{R}^3)$. Let $\varphi(t) := e^{iH_0t} e^{-iHt}\varphi$.

$$\frac{d}{dt}\varphi(t) = e^{iH_0t}\, i(H_0 - H)\, e^{-iHt}\varphi = -i V(t)\varphi(t),$$

where $V := \lambda\tilde{V}$, $V(t) := e^{iH_0t}\, V\, e^{-iH_0t}$.

as always: differentiate!

Integrating: $\varphi(t) = \varphi - i\int_0^t V(t_1)\varphi(t_1)\, dt_1$ (Duhamel formula / interaction picture)

Iterate: take l.H.S. and plug into R.H.S inside the integral, repeat:

$$\varphi(t) = \varphi - i\int_0^t V(t_1)\varphi\, dt + (-i)^2\int_0^t dt_1\, V(t_1)\int_0^{t_1} dt_2\, V(t_2)\,\varphi(t_2)$$

$$= \varphi + \sum_{k=1}^{N} (-i)^k \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \; V(t_1) \cdots V(t_k)\varphi$$

$$+ (-i)^{n+1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \; V(t_1) \cdots V(t_n)\varphi(t_n).$$

If we can show that the last term disappears as $t \to \infty$, then we're in a good situation, because there will be no more full evolution $\varphi(t)$, only the free evolution in $V(t)$.

Last term: $$\left\| \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \; V(t_1) \cdots V(t_n)\varphi(t_n) \right\| \le \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \; \|V\|_\infty \|\varphi\| = \frac{t^n}{n!} \|V\|_\infty^n \|\varphi\| \longrightarrow 0 \quad (n \to \infty).$$

So $$\varphi(t) = \varphi + \sum_{k=1}^{\infty} \underbrace{(-1)^k \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \; V(t_1) \cdots V(t_k)\varphi}_{=:\, \varphi_k(t)} \qquad \text{(Dyson series)}.$$

positive infinity

If $\underbrace{\int_0^\infty dt_1 \cdots \int_0^{t_{k-1}} dt_k \; \| V(t_1) \cdots V(t_k)\varphi \|_{L^2}}_{=:\, c_k} < \infty$, then $\lim_{t\to\infty} \varphi_k(t)$ exists.

If $\sum_{k=1}^{\infty} c_k < \infty$, then we have shown that the convergence of the Dyson series is uniform (i.e. independent of t), implying that we can exchange $t \to \infty$ and the summation of the series, i.e.

$\lim_{t\to\infty} \varphi(t) = \varphi + \sum_{k=1}^{\infty} \lim_{t\to\infty} \varphi_k(t)$, in particular: $\lim_{t\to\infty} \varphi(t)$ exists, as we had to show.

<u>So we have to estimate c_k:</u> Start by change of variables in integral:

$$\| V(t_1) V(t_2) \cdots V(t_k)\varphi \|_{L^2} = \| V e^{-iH_0(t_1 - t_2)} V e^{-iH_0(t_2 - t_3)} \cdots V e^{-iH_0 t_k} \varphi \|_{L^2}$$

$$= \| V e^{-iH_0 s_1} V e^{-iH_0 s_2} \cdots V e^{-iH_0 s_k} \varphi \|_{L^2}$$

where $s_1 = t_1 - t_2$, $s_2 = t_2 - t_3, \ldots, s_k = t_k$.

This is a linear substitution in the integral:

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & \cdots & 1 & -1 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{=: \mathcal{Y}} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-1} \\ t_n \end{pmatrix}$$

$\det \mathcal{Y} = 1 \Rightarrow$ bijective and volume element is unchanged.

So:

$$C_k \leq \int_0^\infty ds_1 \int_0^\infty ds_2 \cdots \int_0^\infty ds_n \; \| V e^{-iH_0 s_1} V e^{-iH_0 s_2} \cdots \varphi \|_{L^2} \, .$$

since $V \in L^1 \cap L^\infty$.

Decompose $\quad V = V_1 \cdot V_2 \, , \quad V_1 = \mathrm{sgn}(V) \, |V|^{1/2} , \; V_2 = |V|^{1/2} \in L^2 \cap L^\infty .$

$$\Rightarrow \| V e^{-iH_0 s_1} V e^{-iH_0 s_2} \cdots V e^{-iH_0 s_n} \varphi \|_{L^2}$$

operator norms

$$= \| V_1 \|_\infty \underbrace{\| V_2 e^{-iH_0 s_1} V_1 \|_{op}}_{} \| V_2 e^{-iH_0 s_2} V_1 \|_{op} \cdots \| V_2 e^{-iH_0 s_k} \varphi \|_{L^2}$$

We estimate this kind of terms now.

$\forall \psi \in \mathcal{H}$:

small-s regime

$$\|V_2 e^{-iH_0 s} V_1 \psi\|_{L^2}$$
$$\leq \|V_2\|_{L^\infty} \|e^{-iH_0 s} V_1 \psi\|_{L^2}$$
$$\leq \|V_2\|_{L^\infty} \|V_1 \psi\|_{L^2} \quad \text{unitarity}$$
$$\leq \|V_2\|_{L^\infty} \|V_1\|_{L^\infty} \|\psi\|_{L^2}$$

large-s regime

$$\|V_2 e^{-iH_0 s} V_1 \psi\|_{L^2}$$
$$\leq \|V_2\|_{L^2} \|e^{-iH_0 s} V_1 \psi\|_{L^\infty}$$
$$\leq \|V_2\|_{L^2} \frac{1}{|s|^{3/2}} \|V_1 \psi\|_{L^1} \quad \text{decay of free evol. in time}$$
$$\leq \|V_2\|_{L^2} \frac{1}{|s|^{3/2}} \|V_1\|_{L^2} \|\psi\|_{L^2}$$

$$\Rightarrow \|V_2 e^{-iH_0 s} V_1\|_{op} \leq \left(\|V_1\|_{L^\infty} \|V_2\|_{L^\infty} + \|V_1\|_{L^2} \|V_2\|_{L^2} \right) \min\Big(\underbrace{1}_{\text{to control } \int_0^1 ds},\ \underbrace{\frac{1}{|s|^{3/2}}}_{\text{to control } \int_1^\infty ds}\Big).$$

Similarly for the last term:

$$\|V_2 e^{-iH_0 s} \varphi\|_{L^2} \leq \Big(\|V_2\|_{L^\infty} \|\varphi\|_{L^2} + \|V_2\|_{L^2} \underbrace{\|\varphi\|_{L^1}}_{\text{ok since } \varphi \in \mathcal{S}(\mathbb{R}^3)} \Big) \min\left(1, \frac{1}{|s|^{3/2}}\right).$$

Thus

$$C_k \leq \int_0^\infty ds_1 \int_0^\infty ds_2 \cdots \int_0^\infty ds_k \ \|V_1\|_\infty \left(\|V_2\|_{L^\infty} \|V_1\|_{L^\infty} + \|V_1\|_{L^2} \|V_2\|_{L^2} \right)^{k-1}$$
$$\times \min\left(1, \frac{1}{|s_1|^{3/2}}\right) \cdots \min\left(1, \frac{1}{|s_k|^{3/2}}\right) \left(\|V_2\|_{L^\infty} \|\varphi\|_{L^2} + \|V_2\|_{L^2} \|\varphi\|_{L^1} \right)$$

V_1, V_2 corresp. to $\sqrt{V}$, so to a $\sqrt{\lambda}$. We count how many powers of V the estimate has and pull out the λ from $V = \lambda \tilde{V}$, yielding:

$$\leq \underbrace{|\lambda|^k \underbrace{C_1(\tilde{V})^k}_{\text{some const. dep. on } \tilde{V}} \Big[\underbrace{\int_0^\infty ds \min\left(1, \frac{1}{|s|^{3/2}}\right)}_{< \infty} \Big]^k}_{=: C_2(\tilde{V})^k}.$$

(in 1D, 2D this would not be integrable!)

If $|\lambda|$ is small enough: $|\lambda| C_2(\tilde{V}) < 1$

$\Rightarrow \sum_k C_k$ is a convergent geometric series;

as discussed before this implies $\operatorname{ran} \Omega_+ = L^2(\mathbb{R}^3)$. ∎