Solutions to Assignment 3

Problem 3: By Cook's argument (see Thm. on existence of wave operators in lecture notes) it is sufficient to show

$$\int_{\Lambda}^{\infty} \| V e^{-iH_0 t} \varphi \| dt < \infty$$

$\Lambda \leftarrow$ any fixed number ($<\infty$!)

with φ from a dense subspace X of $L^2(\mathbb{R}^n)$.

Take $X := \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \}$.

Let $\varphi \in X$. Then $\exists\, \varepsilon > 0$ such that $\operatorname{supp} \hat{\varphi} \subset \{ p : |p| \geq \varepsilon \}$.

Since $|\nabla\omega(p)| = \dfrac{|p|}{\sqrt{p^2+1}}$ is increasing w.r.t. $|p|$ we get

$$|\nabla\omega(p)| \geq \frac{\varepsilon}{\sqrt{\varepsilon^2+1}} =: 2\delta, \quad \forall p \in \operatorname{supp} \hat{\varphi}.$$

(Interpretation: in classical mechanics $|\nabla\omega(p)|$ corresponds to the velocity of a particle. In the set X, every $\hat{\varphi}$ has a lower bound on the momenta contained in it, which gives us a lower bound on $|\nabla\omega|$ – the "minimal velocity" at which free particles fly away.)

Decompose: $V = V_2^t + V_\infty^t$,

classically no particle expected in this region

$$V_2^t(x) = V(x)\, \chi_{\{x : |x| \leq t\cdot\delta\}}(x)$$

$$V_\infty^t(x) = V(x)\, \chi_{\{x : |x| > t\cdot\delta\}}(x)$$

region where the particle is expected

Bounds: $\| V_2^t \|_{L^2} \leq \| V \|_{L^\infty} \left(\int_{|x| \leq t\cdot\delta} 1 \, d^n x \right)^{1/2} = C t^{n/2}$

$$\| V_\infty^t \|_\infty \leq \frac{\text{const.}}{|t|^\mu} \qquad \text{for } \delta t > R.$$

So $\| V_2^t e^{-iH_0 t} \varphi \|_{L^2} \leq \| V_2^t \|_{L^2} \| \chi_{\{x : |x| \leq \delta t\}} e^{-iH_0 t} \varphi \|_{L^\infty}$

$\boxed{\text{using stat. phase (check assumptions!)}}$ $\leq \text{const. } t^{n/2} \dfrac{C_m}{(1 + |t|)^m} \quad \left(\begin{matrix}\text{any} \\ m \in \mathbb{N}\end{matrix}\right).$

$$\| V_\infty^t e^{-iH_0 t} \varphi \|_{L^2} \leq \| V_\infty^t \|_{L^\infty} \| e^{-iH_0 t} \varphi \|_{L^2}$$

$$\leq \frac{\text{const.}}{|t|^\mu} \| \varphi \|_{L^2} \qquad \left(\text{for } t > R/\delta\right).$$

Since $\mu > 1$: pick $m > \frac{n}{2} + 1$, then both bounds are integrable at infinity, so Cook's argument is complete. ∎

PROBLEM 4: (Magnetic Schrödinger operator)

$$H_A := -\Delta + \vec{A}^2 - 2i \vec{A} \cdot \vec{\nabla}.$$

To apply Kato-Rellich, first check domains:
We have to show $D(\vec{A}^2 - 2i\vec{A} \cdot \vec{\nabla}) \supseteq H^2(\mathbb{R}^3)$.

We have $\displaystyle\int dx \, |\vec{A}^2 \psi|^2 \leq \underbrace{\|\psi\|_{L^\infty}^2}_{\leq \|\psi\|_{H^2}^2 < \infty} \underbrace{\int dx \, |A|^4}_{< \infty}$ since $\vec{A} \in H^1$ $\Rightarrow \vec{A} \in L^2 \cap L^6$ $\Rightarrow \vec{A} \in L^4$.

check this!

So $\quad \psi \in H^2 \Rightarrow \int dx\, |\vec{A}^2 \psi|^2 < \infty \overset{\text{def.}}{\Leftrightarrow} \psi \in D(\vec{A}^2)$

Furthermore $\int dx\, |\vec{A}\cdot\vec{\nabla}\psi|^2 < \infty$ where $\vec{\nabla}\psi$ is well-def. for $\psi \in H^2$, so also $H^2 \subseteq D(\vec{A}\cdot\vec{\nabla})$. (Estimate similar to what we do below, no details here).

Now we have to check:

$$\|(\vec{A}^2 - 2i\,\vec{A}\cdot\vec{\nabla})\psi\|_{L^2} \leq a\,\|\Delta\psi\|_{L^2} + b\,\|\psi\|_{L^2}$$

where $a < 1$ and b arbitrary.

Separately by triangle inequality:

$$\|\vec{A}^2\psi\|_{L^2}^2 = \int dx\, |\vec{A}^2\psi|^2 \leq \int dx\, |\vec{A}|^4\, |\psi|^2$$

$$\underset{\substack{\text{Hölder} \\ (p=3,\ q=\frac{3}{2})}}{\leq} \left(\int dx\, |\vec{A}|^{4\cdot 3/2}\right)^{2/3} \left(\int dx\, |\psi|^{2\cdot 3}\right)^{1/3}$$

$$= \underbrace{\|\vec{A}\|_{L^6}^4}_{=\text{const.}}\ \underbrace{\|\psi\|_{L^6}^2}_{\leq \text{const.}\ \|\nabla\psi\|_{L^2}^2}$$

$$\leq \text{const.}\ \langle \psi, -\Delta\psi \rangle$$

$$\leq \text{const.}\ \|\psi\|\frac{1}{\varepsilon}\cdot \varepsilon\,\|-\Delta\psi\|$$

$$\Rightarrow \quad \|\vec{A}^2\psi\|_{L^2} \leq \text{const.}\ \sqrt{\|\psi\|\frac{1}{\varepsilon}}\ \sqrt{\varepsilon\,\|-\Delta\psi\|}$$

$$\leq \text{const.}\left(\varepsilon\,\|-\Delta\psi\| + \frac{1}{\varepsilon}\|\psi\|\right). \qquad (i)$$

Now we have to similarly estimate $\|-2i\vec{A}\cdot\vec{\nabla}\|_{L^2}^2$:

$$\|\vec{A}\cdot\vec{\nabla}\psi\|_{L^2}^2 = \int dx\, |\vec{A}\cdot\vec{\nabla}\psi|^2 \leq \int dx\, |\vec{A}|^2\, |\vec{\nabla}\psi|^2$$

$$\overset{\text{Hölder}}{\leq} \underbrace{\left(\int|\vec{A}|^6 dx\right)^{1/3}}_{\leq \text{ const.} < \infty} \left(\int|\nabla\psi|^3 dx\right)^{2/3}$$

since by Sobolev $\|\cdot\|_{L^6} \leq \text{const.} \|\cdot\|_{H^1}$ and $\vec{A} \in H^1$ by assumption

Idea: separate off the $\vec{A}$

Idea: get to L^6, because can be controlled with one derivative

Hölder $p=4$, $\frac{1}{p}+\frac{1}{q}=1$, $q=\frac{4}{3}$.

$$\leq \text{const.} \left(\int|\nabla\psi|^{3/2}\,|\nabla\psi|^{3/2} dx\right)^{2/3}$$

$$\leq \text{const.} \left(\left(\int|\nabla\psi|^{3/2\cdot 4} dx\right)^{1/4}\left(\int|\nabla\psi|^{3/2\cdot\frac{4}{3}} dx\right)^{3/4}\right)^{2/3}$$

$$= \text{const.} \left(\|\nabla\psi\|_{L^6}\cdot\|\nabla\psi\|_{L^2}\right)$$

$$\Rightarrow \|\vec{A}\cdot\vec{\nabla}\psi\|_{L^2} \leq \text{const.} \sqrt{\|\nabla\psi\|_{L^6}}\sqrt{\|\nabla\psi\|_{L^2}}$$

$$\leq \text{const.} \left(\varepsilon\|\nabla\psi\|_{L^6} + \frac{1}{\varepsilon}\|\nabla\psi\|_{L^2}\right)$$

$$\overset{\text{Sobolev}}{\leq} \text{const.} \left(\varepsilon\|-\Delta\psi\|_{L^2} + \frac{1}{\varepsilon}\|\nabla\psi\|_{L^2}\right).$$

remains to estimate this!

$$\|\nabla\psi\|_{L^2}^2 = \sqrt{\langle\psi, -\Delta\psi\rangle} \leq \sqrt{\frac{1}{\delta}\|\psi\|}\sqrt{\delta\|-\Delta\psi\|}$$

$$\leq \frac{1}{\delta}\|\psi\| + \delta\|-\Delta\psi\|.$$

$$\Rightarrow \|\vec{A}\cdot\vec{\nabla}\psi\|_{L^2} \leq \text{const.} \left(\varepsilon\|-\Delta\psi\|_{L^2} + \frac{1}{\varepsilon}\delta\|-\Delta\psi\| + \frac{1}{\varepsilon\delta}\|\psi\|\right). \quad \text{(ii)}$$

Sum up (i) and (ii), then choose first ε, then δ very small, so that you get the estimate with $a<1$ as needed for Kato-Rellich. ∎