Trotter's product formula and the BCH formula

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1 Trotter's product formula

We will now show Trotter's product formula, following the proof of [Simon, 1979].

Theorem 1 (Trotter's product formula)**.** *Let H be a Hilbert space. Assume A, B self-adjoint on H, and that $A + B$ with domain $D(A) \cap D(B)$ is also self-adjoint. Then the following holds*

$$\exp(it(A+B)) = \text{s-lim}_{n\to\infty}(\exp(itA/n)\exp(itB/n))^n \tag{1}$$

Proof. We define the one-parameter unitary groups

$$S_t = \exp(it(A+B)),\ V_t = \exp(itA) \tag{2}$$

$$U_t = \exp(itB),\ W_t = V_t U_t \tag{3}$$

and for all $\xi \in H$ define $\xi_t = S_t\xi$. We start by noting that we have the equality

$$S_{t/n}^n - W_{t/n}^n = \sum_{j=0}^{n-1} W_{t/n}^j (S_{t/n} - W_{t/n}) S_{t/n}^{n-j-1}$$

since we recognize the right hand side as simply a telescoping expansion of the left hand side. Now we may write

$$\left|\left|(S_{t/n}^n - W_{t/n}^n)\xi\right|\right| = \left|\left|\sum_{j=0}^{n-1} W_{t/n}^j (S_{t/n} - W_{t/n}) S_{t/n}^{n-j-1}\xi\right|\right| \leq \sum_{j=0}^{n-1}\left|\left|(S_{t/n} - W_{t/n}) S_{t/n}^{n-j-1}\xi\right|\right| \tag{4}$$

$$\leq n \sup_{0\leq s\leq t}\left|\left|(S_{t/n} - W_{t/n})\xi_s\right|\right| \tag{5}$$

Now consider vectors lying in $D(A) \cap D(B)$. We have that $\lim_{s\to 0}\frac{S_s - 1}{s} = i(A+B)$ here. We also wish to calculate the derivative of W_s as we just did for S_s

$$\begin{aligned}\frac{W_s - 1}{s}\varphi &= V_s iB\varphi - V_s iB\varphi + \frac{V_s U_s - 1}{s}\varphi \\ &= V_s iB\varphi + V_s\left(\frac{U_s - 1}{s} - iB\right)\varphi + \frac{V_s - 1}{s}\varphi\end{aligned}$$

We see that this expression tends to $iB\varphi + iB\varphi - iB\varphi + iA\varphi = iB\varphi + iA\varphi$ as $s \to 0$. As $S_0 = W_0 = 1$ this gives us

$$\lim_{n\to\infty} \left|\left|n(S_{t/n} - W_{t/n})\right|\right| = \lim_{n\to\infty} \left|\left|\frac{(S_{t/n} - 1) - (W_{t/n} - 1)}{1/n}\right|\right| = 0 \tag{6}$$

everywhere in $D(A) \cap D(B)$. Since $A + B$ is self-adjoint it is closed, therefore $D(A) \cap D(B)$ becomes a Banach space when equipped with the graph norm. We have just shown that $n(S_{t/n} - W_{t/n})$ are bounded for every n, with $\sup_{n\in\mathbb{N}}\{||n(S_{t/n} - W_{t/n})\varphi||\} < \infty$. Then the uniform boundedness principle gives us that $n(S_{t/n} - W_{t/n})$ is uniformly bounded as $||n(S_{t/n} - W_{t/n})\varphi|| \leq C\,||\varphi||$, viewed as an operator from $D(A) \cap D(B)$ equipped with the graph norm to H equipped with the usual norm. This gives us that the convergence in Equation (6) is uniform on compact subsets of $\mathbb{R}$. We note that the mapping $\hat{\xi} : s \mapsto \xi_s$ is continuous, therefore the image of a compact set is compact, in particular the image of the interval $[0, t]$, which together with our previous results imply that as $n \to \infty$ Equation (5) goes to zero, which shows the theorem.

□

Another essential result relating the product of the exponentials with the exponential of the sum is the Baker Campbell Hausdorff formula. We start with a minor lemma

Lemma 2. *Let X be a Banach space and consider $L(X)$. Let $A, B \in L(X)$ satisfy that $[A, [A, B]] = [B, [A, B]] = 0$. Then $\exp(tB)A\exp(-tB) = A - t[A, B]$.*

Proof. Consider the Taylor expansion of $f(t) = \exp(tB)A\exp(-tB)$.

$$\begin{aligned}f'(t) &= \exp(tB)BA\exp(-tB) + \exp(tB)(-AB)\exp(-tB) \\ &= \exp(tB)[B, A]\exp(-tB) \\ &= -[A, B]\end{aligned}$$

Thus we have $f(t) = f(0) + tf'(t) = A - t[A, B]$, as desired. □

We can use this lemma to prove the Baker-Campbell-Hausdorff formula.

Lemma 3. *For $A, B \in L(X)$ where X is a Banach space, and $[A, [A, B]] = [B, [A, B]] = 0$ we* *have the formula*

$$\exp(A)\exp(B) = \exp\left(\frac{1}{2}[A,B]\right)\exp(A+B)$$

Proof. We see that the formula we wish to derive is equivalent to

$$\exp(A)\exp(B)\exp(-(A+B)) = \exp\left(\frac{1}{2}[A,B]\right)$$

Defining the function $f(t) = \exp(tA)\exp(tB)\exp(-t(A+B))$ we need merely check that its derivative is $t[A,B]f(t)$, as this will imply that $f(t) = \exp\left(\frac{1}{2}t^2[A,B]\right)$, by uniqueness of solutions of ODE's.

$$\begin{aligned} f'(t) &= \exp(tA)[A\exp(tB)\exp(-t\{A+B\}) \\ &\quad + \exp(tB)B\exp(-t\{A+B\}) + \exp(tB)(-\{A+B\})\exp(-t\{A+B\})] \\ &= \exp(tA)[A,\exp(tB)]\exp(-t(A+B))] \\ &= \exp(tA)(A - \exp(tB)A\exp(-tB))\exp(tB)\exp(-t(A+B)) \\ &= \exp(tA)(A - (A - t[A,B]))\exp(-tB)\exp(-t(A+B)) \\ &= t[A,B]f(t) \end{aligned}$$

Thus we have that $f(t) = \exp\left(\frac{1}{2}t^2[A,B]\right)$, and evaulating in $t = 1$ gives us the desired result. □

References

[Grubb, 2009] Grubb, G. (2009). Distributions and operators, volume 252 of Graduate Texts in Mathematics. Springer, New York.

[Simon, 1979] Simon, B. (1979). Functional Integration and Quantum Physics. Academic Press.