Fermi Coherent States with and without Grassmann Numbers

-1) $\longrightarrow$ Motivation: p. 0.

0) General setting:

For simplicity: # deg. of freedom $< \infty$: (i.e. one-particle H.S. has dim $< \infty$)

Quick. One-fermion H.S.: $L^2(X) \simeq \mathbb{C}^{|X|}$, where $|X| < \infty$. (e.g. X = finite lattice or finite set of momenta)

n fermions: $\mathcal{F}_n = \{ f \in L^2(X^n) : f \text{ antisymm.} \}$

with inner product:

$$\langle f, g \rangle = \int_{X^n} dx_1 \ldots dx_n \, \overline{f(x_1, \ldots x_n)} \, g(x_1, \ldots x_n)$$

counting measure, i.e. $\int_X dx = \sum_{x \in X}$.

Fock space: $\mathcal{F} = \bigoplus_{n=0}^{|X|} \mathcal{F}_n$: sum up to $|X|$ b/c antis. of $> \dim(L^2(X))$-many states vanishes.

States: Sequences $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots)$.

Vacuum: $\Omega = |0\rangle = (1, 0, 0, \ldots)$

Annihilation op.: For $u \in \mathcal{F}_n$:

$$(a(x) u)(x_1, \ldots x_{n-1}) = \sqrt{n} \, u(x, x_1, \ldots x_{n-1}).$$

Creation op.:

sum over all perm. of arguments, but put a ± 1 coefficient for even/odd perms

$$(a^*(x) u)(x_1, \ldots x_{n+1}) = (\delta_x \wedge u)(x_1, \ldots x_{n+1})$$

$$= \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} (-1)^{j-1} \delta_x(x_j) u(x_1, \ldots x_{j-1}, x_{j+1}, \ldots x_{n+1}).$$

CAR: $[a(x), a^*(y)]_+ = a(x) a^*(y) + a^*(y) a(x)$

$$= \delta_x(y),$$

others vanish:

$$[a(x), a(y)]_+ = [a^*(x), a^*(y)]_+ = 0.$$

$a(x)\Omega = 0$ (vacuum has no particles $\rightsquigarrow$ destroyed by $a(x)$.)

(2)

Can we find $\Psi(f) \in \mathcal{F}$
($f \in L^2(X)$: parameter)
s.th. $\underbrace{a(x)\Psi(f) = f(x)\Psi(f)}_{\text{eigenvalue equation}}$?

If there was $\Psi(f) \in \mathcal{F}$ with
$a(x)\Psi(f) = f(x)\Psi(f) \; \forall x \in \mathbb{R}^3$, then
$f(x)f(y)\Psi(f) = -f(y)f(x)\Psi(f)$!

... Let's go for Grassmann or find a workaround...

Proposed solutions:

f as anti-commutative eigenvalues → Grassmann numbers → approach 1 (more common)

relax eigenvalue equation → $\langle \Psi(f), a^*(x)a(x)\Psi(f)\rangle = \overline{f(x)}f(x)$. → approach 2

slow down

1) Grassmann approach:

Brute force: tensor Fock space with anticommutative algebra to allow for anticommutative coefficients.

Let's introduce the anticomm. algebra (Grassmann a.) first:

$\mathcal{D}$:= vector space freely generated by

$$\underbrace{\{\phi(x) : x \in X\} \cup \{\phi(x)^* : x \in X\}}_{} .$$

I will also use the notation: $= \{\xi_1, \dots \xi_D\}$, $D = 2|X|$.

Warning: $\phi(x)$, $\phi(x)^*$ are just indep. basis vectors labelled with „x" and „*".
„*" does not mean complex conjugation, „x" is just an index! We just force notation to look like complex numbers!

Grassmann algebra:

$$\mathcal{G} := \bigoplus_{n=0}^{\dim \mathcal{V}} \Lambda^n \mathcal{V}.$$ Antisymmetrized tensor product.

Each element of $\mathcal{G}$ can be written as

$$a(\xi) = \sum_{n=0}^{D} \sum_{\substack{\vec{\imath} \in \{1,\dots D\}^n \\ 1 \le i_1 < i_2 \dots < i_n \le D}} a_{\vec{\imath}}^{(n)} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}.$$

(try not to confuse with annih. operator)

antisymm. ⊗-prod. „Λ" left away

sum only over even n ↳ $a(\xi)$ is even.

sum over odd n ↳ $a(\xi)$ is odd.

Then $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$.

$\mathcal{G}_+$: even subspace; $\mathcal{G}_-$: odd subspace.

Lemma:

- Even elements commute with everything:
  $a(\xi) \in \mathcal{G}_+$, $b(\xi) \in \mathcal{G}$ ← arbitrary
  $\Rightarrow a(\xi) b(\xi) = b(\xi) a(\xi)$.
- Odd e. with odd e. anticommute:
  $a(\xi), b(\xi) \in \mathcal{G}_-$
  $\Rightarrow a(\xi) b(\xi) = -b(\xi) a(\xi)$.

Ex.: $a = \xi_1 \xi_2$, $b = \xi_3$.

Then

$$ab = \xi_1 \xi_2 \xi_3 = \xi_1(-\xi_3 \xi_2) = (-1)^2 \xi_3 \xi_1 \xi_2 = ba.$$

Formal Grassmann calculus: (no limits, no measure theory)

- Integration: Linear map $\int d\xi_D \cdots d\xi_1 : \mathcal{G} \to \underbrace{\mathbb{C}}$

can be used to "integrate out" Grassmann numbers

with:
- $\int d\xi_D \cdots d\xi_1 \big|_{\bigoplus_{n=0}^{D-1} \Lambda^n V} = 0$
- $\int \underbrace{d\xi_D \cdots \underbrace{d\xi_1 \; \xi_1} \cdots \xi_D} = 1.$

notice the order.

If differing $\rightsquigarrow$ possibly find minus signs.

not needed, leave away

- Derivative: $\frac{\partial}{\partial \xi_j} : \mathcal{G} \to \mathcal{G}$
  - $\frac{\partial}{\partial \xi_j} \xi_{i_1} \cdots \xi_{i_k} \xi_j \xi_{i_{k+1}} \cdots \xi_{i_n} = (-1)^k \xi_{i_1} \cdots \xi_{i_k} \xi_{i_{k+1}} \cdots \xi_{i_n}$

    i.e. to be commuted to ξ_j, $\frac{\partial}{\partial \xi_j} \xi_j = 1$.
  - $\frac{\partial}{\partial \xi_j} \xi_{i_1} \cdots \xi_{i_n} = 0$ if all $i_k \neq j$.

Lemma: (Change of variables)

Let $A \in GL(n)$. Let $\tilde{\xi}_k := \sum_{j=1}^{D} A_{kj} \xi_j$, "$\tilde{\xi} = A\xi$".

Then

$$\int \underbrace{a(\tilde{\xi})} \, d\xi = \int \underbrace{a(\xi)} \, d\xi \cdot \det(A).$$

lin. comb. of products of the transformed var., $\tilde{\xi}$

$d\xi_1 \ldots d\xi_D$ integrations over all Grassmann var.

same lin. comb. but $\tilde{\xi}$s replaced by ξs.

No proof, just want to point out that change of var. is possible.

<u>We now exted Fock space to allow for Grassmann coefficients.</u>

$\Gamma := \{$linear operators on $\mathcal{F}$ given through creation and annihilation op.$\}$ (in finite dim. setting these are all)

$\Gamma_\pm :=$ subspaces of Γ generated by products of even resp. odd numbers of cr./ann. op.

<u>Ex.:</u> $\int dx\, dy\; a_x^* a_y^* a_x a_y v(x-y) \in \Gamma_+$.

We define an associative, distributive mult. on $\mathcal{G} \otimes \Gamma$:

For $A, B \in \Gamma$, $\psi, \sigma \in \mathcal{G}$, let

$$A = \underset{\Gamma_+}{\underset{\cap}{A_+}} + \underset{\Gamma_-}{\underset{\cap}{A_-}}, \quad \sigma = \underset{\mathcal{G}_+}{\underset{\cap}{\sigma_+}} + \underset{\mathcal{G}_-}{\underset{\cap}{\sigma_-}}.$$

Set

$$[\psi \otimes A][\sigma \otimes B] := \underbrace{\psi\sigma_+}_{\text{prod. in } \mathcal{G}} \otimes \underbrace{A\, B}_{\text{prod. in } \Gamma\text{, space of op.}}$$
$$+ \psi\sigma_- \otimes A_+ B$$
$$- \psi\sigma_- \otimes A_- B.$$

Why? You want to mult. ψ with σ and B with A. But to mult. σ to ψ, A and σ have to be „commuted".

But A and σ live in different spaces! What does it mean to commute them? We define it to give a „$-$" if σ and A are <u>odd</u> elements.

(As before, only „odd" with „odd" gives a minus.)

$\in \mathcal{G}_-$ $\quad \in \Gamma_-$ (annihil.)

Ex.: $\Phi(x) \otimes a(y) = [\Phi(x) \otimes 1\!\!1][1\!\!1 \otimes a(y)]$
$$= -[1\!\!1 \otimes a(y)][\Phi(x) \otimes 1\!\!1].$$

Put more casually:
$$\Phi(x)\, a(y) = -\, a(y)\, \Phi(x). \qquad (*)$$

This is all you have to remember: Cr./ann. op. anticommute with Grassmann numbers.

Ex.: $\underset{\mathcal{G}_\psi}{\phi(x)}\, a(y) \cdot \xi \underset{\mathcal{F}_\psi}{\psi}$
$$\overset{(-1)}{=} -\underbrace{\underbrace{\phi(x)\,\xi}_{\in \mathcal{G}}\ \underbrace{a(y)\,\psi}_{\in \mathcal{F}}}_{\in \mathcal{G} \otimes \mathcal{F}}.$$

In natural way, $\mathcal{G} \otimes \Gamma$ acts on $\mathcal{G} \otimes \mathcal{F}$.

Fermionic Coherent States:

Let $\Phi^* \cdot \Phi := \int_X dx\ \Phi(x)^* \Phi(x)$. (Sum)

$$\text{CS: } \Psi_\Phi := e^{-\Phi^* \cdot \Phi/2}\, e^{-\int_X dx\ \Phi(x)\, a^*(x)}\, \Omega$$
$$= e^{-\Phi^* \cdot \Phi/2} \prod_{x \in X} \left(1 - \Phi(x)\, a^*(x)\right) \Omega$$

because $\int_X dx = \sum_{x \in X}$

- Notice that exp has only first order b/c square of anticommuting objects $= 0$.
- Notice that order of the x is not important b/c $\Phi(x)\, a^*(x)$ commutes with $\Phi(y)\, a^*(y)$ (even number of minuses).

Lemma: (Eigenvalue equ.)

$$a(x)\, \Psi_\Phi = \Phi(x)\, \Psi_\Phi \quad \forall x \in X.$$

Ex.: Single-mode: $e^{-\phi^*\phi/2}$ } even, commutes with every thing, just a pre-factor.

$$\begin{aligned} a\Psi_\phi &= a(1-\phi a^*)\Omega \\ &= \underbrace{a\,\Omega}_{=0} + \phi \underbrace{a a^* \Omega}_{=\Omega} \\ &= \phi\,\Omega \\ &= \phi(1-\phi a^*)\Omega \quad (\phi^2=0) \\ &= \phi\,\Psi_\phi . \end{aligned}$$

Scalar product on $\mathcal{G}\otimes\mathcal{F}$: ? see p. 7.1

I have some conj. for that, but from physics literature and inconsistent (?).

My references have not been clear on that, we will now introduce a formal dual vector:

$$\langle\Psi_\phi| \overset{\text{def.}}{:=} \langle 0| \prod_{x\in X} (1 - a(x)\phi(x)^*)\, e^{-\phi^*\phi/2}$$
(formally)

← note again: this is not a conjugation.

Then we get:

can be extended to $\varphi\in\mathcal{G}\otimes\mathcal{F}$.

„Res. of identity"

Lemma: Let $\varphi\in\mathcal{F}$. Then

$$\int\Big(\prod_{x\in X} d\phi(x)^*\, d\phi(x)\Big)\, \underbrace{\Psi_\phi}_{\text{coherent state}}\overbrace{\langle\underbrace{\Psi_\phi}_{\in\,\mathcal{G}\otimes\mathcal{F}}, \underbrace{\varphi}_{\in\,\mathcal{F}}\rangle}^{\in\,\mathcal{G}} = \varphi .$$

Int. over all Grassmann generators

How is this defined? Will not write def. explicitly, we will do two examples and you'll see that there is a natural way [for calculating this] from the context.

~~Then $\int d\phi^* d\phi\, |\Psi_\phi\rangle\langle\Psi_\phi|\varphi\rangle$~~

~~$= \int e^{-\phi^*\cdot\phi}(1-\phi a^*)\Omega\, \langle 0|(1-a\phi^*)\,|\varphi\rangle\, d\phi^*\, d\phi$~~

~~$= \int e^{-\phi^*\cdot\phi}\, (|0\rangle - \phi|1\rangle)(\langle 0| - \langle 1|\phi^*)$~~

Skalarprodukt: (Conj.)

On $\mathcal{G}$:

Let $f(\phi) = f_0 + f_1 \phi$, $\bar{f}(\phi) = \bar{f}_0 + \bar{f}_1 \phi$

$g(\phi) = g_0 + g_1 \phi$, $g(\phi^*) = g_0 + g_1 \phi^*$.

Define (physicist's ...) // yest. afternoon

$$\langle f, g \rangle_{\mathcal{G}} := \int d\phi^* d\phi \, e^{-\phi^* \phi} \, \bar{f}(\phi) \, g(\phi^*)$$

$$= \bar{f}_0 g_0 + \bar{f}_1 g_1 ,$$

So esp. $\langle f, f \rangle_{\mathcal{G}} = |f_0|^2 + |f_1|^2$:

non-neg., non-degenerate.

Proposal: If $f \otimes a \in \mathcal{G} \otimes \mathcal{F}$:

$$\langle f \otimes a, f \otimes a \rangle := \int d\phi^* d\phi \, e^{-\phi^* \phi} \, \bar{f}(\phi) \, f(\phi^*) \langle a, a \rangle$$

$$= (|f_0|^2 + |f_1|^2) \, \|a\|^2 :$$

non-neg., non-dege.

But shouldn't $\langle f \overset{*}{\otimes} a, f \otimes a \rangle$ (± 1) ?

Ex. 1: Single-mode:

$$\int d\phi^* d\phi \ \Psi_\phi \langle \Psi_\phi |$$

$$= \int d\phi^* d\phi \ e^{-\phi^*\phi} (1 - \phi a^*)|0\rangle\langle 0|(1 - a\phi^*)$$

$$= \int d\phi^* d\phi (1 - \phi^*\phi)(1 - \phi a^*)|0\rangle\langle 0|(1 - a\phi^*)$$

$$= \int d\phi^* d\phi \ (+\phi a^*)|0\rangle\langle 0|(+a\phi^*)$$

$$+ \int d\phi^* d\phi \ (\underbrace{-\phi^*\phi}_{(-1)}) \ \overset{a^* a:\ \text{even}}{|0\rangle\langle 0|}$$

$$\left\{ \phi^2 = \phi^{*2} = 0 \right.$$

$$\left\{ \begin{array}{l} \int d\phi^* d\phi \ \phi^* = 0 \\ \int d\phi^* d\phi \ \phi = 0 \\ \int d\phi^* d\phi \ 1 = 0 \\ \int d\phi^* d\phi \ \phi\phi^* = 1 \end{array} \right.$$

$$= \int d\phi^* d\phi \ \phi\phi^* \ a^*|0\rangle\langle 0|a$$

$$+ \underbrace{\int d\phi^* d\phi \ \phi\phi^*}_{=1} \ |0\rangle\langle 0|$$

$$= |1\rangle\langle 1| + |0\rangle\langle 0| = \mathbb{1}.$$

Notice: $|1\rangle\phi = \overset{(-1)}{a^* |0\rangle \phi} = -\phi|1\rangle.$

(for these formal calc.)

Ex. 2: Slater det.:

Let $a^*(u) = \int dx \ a_x^* u(x), \quad u \in L^2(X, \mathbb{C})$

single-part. wavefct.

Then $u_1 \wedge \dots \wedge u_n = a^*(u_1) \cdots a^*(u_n)\Omega.$

Then

$$\langle \Psi_\phi, u_1 \wedge \dots \wedge u_n \rangle$$

$$= \langle a(u_n) \cdots a(u_1)\Psi_\phi, \Omega \rangle$$

$$= (u_n \cdot \phi) \cdots (u_1 \cdot \phi) \langle \Psi_\phi, \Omega \rangle$$

$$= (u_n \cdot \phi) \cdots (u_1 \cdot \phi) \ e^{-\phi^*\phi/2} \in \mathcal{G}.$$

where $u_n \cdot \phi = \int dx \, u_n(x)\phi(x) \in \mathcal{G}_-$.

Extension to infinitely many degrees of freedom possible, but more technical.

I rather want to use the left-over time to mention the non-Grassmannian approach:

2) Non-Grassmann approach:

Haven't seen much material about that, mainly the motivation and Def. only, so that's what I will present now.

Maybe this is just a complicated way of stating s.th. obvious, I didn't have much time to think about it.

Consider Grassmann coherent state with inf. d.o.f., x as continuous index" $\Leftrightarrow$ inf. many indep. Grassmann generators $\rightsquigarrow$ exp. series does not terminate:

$$\Psi_\phi = \underbrace{e^{-\phi^*\phi/2}}_{N} \; \underbrace{e^{\int dx \, a^*(x)\,\phi(x)}}_{\sum}$$

$$= N \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\int dx \, a^*(x)\,\phi(x) \Big)^n \Omega$$

can be pulled out of product to the end, b/c $a^*(x)\phi(x)$ are even.

$$= N \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n \, \underbrace{a^*(x_1) \cdots a^*(x_n)\Omega}_{=: |x_1, \dots x_n\rangle_A} \; \phi(x_n) \cdots \phi(x_1).$$

antisymm.

We can not make $\phi(x_n) \cdots \phi(x_1)$ $\mathbb{C}$-numbers $\rightsquigarrow$ so symm. w.r.t. permut b/c $|x_1 \dots x_n\rangle_A$ antisymm. But: make $|x_1 \dots x_n\rangle_A$ symm.

We introduce an arbitrary ordering (so that we can say what standard ordering is ...) of points $x = (x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3$,

e.g. $x < y \;:\Leftrightarrow$ (i) $x^{(1)} < y^{(1)}$ or

(ii) $x^{(1)} = y^{(1)} \wedge x^{(2)} < y^{(2)}$ or

(iii) $x^{(1)} = y^{(1)} \wedge x^{(2)} = y^{(2)} \wedge x^{(3)} < y^{(3)}$.

Define a sign function:

$$\sigma_n(x_1, x_2, \dots x_n) := \begin{cases} \bullet\ \mathrm{sgn}(\pi) \text{ where } \pi \in \mathcal{S}_n \text{ brings } x_1, \dots x_n \text{ to standard order} \\ \bullet \text{ if any } x_i = x_j : \ = 0. \end{cases}$$

Now define fully symmetric „states"

$$|x_1, \dots x_n\rangle_S := \sigma_n(x_1, \dots x_n)\, |x_1, \dots x_n\rangle_A .$$

Remark: Since $\sigma_n^2 = 1$ for distinct arguments:

This can be inverted: $|x_1 \dots\rangle_A = \sigma(x_1 \dots)\, |x_1 \dots\rangle_S$.

Def: Commuting variable coherent states:

$$|\phi\rangle := \tilde{N} \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots \int dx_n\, \phi(x_1) \dots \phi(x_n)\, |x_1, \dots x_n\rangle_S$$

Normalization.

where $\phi \in L^2(\mathbb{R}^3, \underline{\underline{\mathbb{C}}})$. — no Grassmann numbers anymore, antisymmetry put into $\sigma_n(\dots)$!

Properties:

- Similar to bosons save that no contrib. for coinciding points.
- Coinc. points have vanishing Lebesgue measure $\rightsquigarrow$ just like bosonic case:

$$\tilde{N} = e^{-\int |\phi|^2/2} = e^{-\|\phi\|^2/2}$$

C.S. $\rightarrow$ $$\langle \phi, \phi' \rangle_{Fock} = e^{-\|\phi\|^2/2}\, e^{-\|\phi'\|^2/2}\, e^{\langle \phi, \phi' \rangle_{L^2(\mathbb{R}^3)}}$$

- $|\phi\rangle$ are <u>not</u> eigenstates of $a(x)$, but
- $\langle \phi | a^*(x)\, a(x) | \phi \rangle = \overline{\phi(x)} \cdot \phi(x)$, which has been useful to relate classical and quantum expressions. (non-rigorous papers?)

There is much more to tell, and also for we many open questions, but we want to have dinner.