Construction of First Integrals for the Hénon–Heiles model by the method of Poincaré [work in progress]

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1 The Method of Poincaré

Consider a Hamilton function, with positions $x = (x_1, \ldots, x_n)$ and momenta $y = (y_1, \ldots, y_n)$, consisting of a quadratic part and an interaction (with a coupling constant $\varepsilon \in \mathbb{R}$ that we think of as small but will eventually take it as $\varepsilon = 1$):

$$H(x, y) = H_0(x, y) + \varepsilon H_1(x, y)$$

where

$$H_0(x, y) = \frac{1}{2} \sum_{l=1}^{n} \omega_l (x_l^2 + y_l^2) \, , \qquad \omega_l > 0 \, .$$

We assume that H_1 is a homogeneuous polynomial of degree 3 as in the Hénon–Heiles model, although this can be easily generalized. In the Hénon–Heiles model, the number of particles is $n = 2$ and they move in one–dimensional space $\mathbb{R}$.

Diagonalization of the Hamiltonian H_0 generating a linear evolution By transforming to complex coordinates

$$\left. \begin{aligned} x_l &= \frac{1}{\sqrt{2}}(\xi_l + i\eta_l) \\ y_l &= \frac{i}{\sqrt{2}}(\xi_l - i\eta_l) \end{aligned} \right\} \quad \longleftrightarrow \quad \left\{ \begin{aligned} \xi_l &= \frac{1}{\sqrt{2}}(x_l - iy_l) \\ \eta_l &= \frac{-i}{\sqrt{2}}(x_l + iy_l) \end{aligned} \right. \tag{1.1}$$

we get

$$H_0 = \sum_{l=1}^{n} \omega_l I_l \, , \qquad I_l := i\xi_l \eta_l \, .$$

The linear Hamiltonian system defined by H_0 has n independent first integrals, namely

$$I_l = \frac{1}{2}(x_l^2 + y_l^2) = i\xi_l \eta_l \, , \qquad l \in \{1, 2, \ldots, n\} \, .$$

Poisson brackets are easy to compute also in complex coordinates:

Lemma 1.1 (Poisson bracket in complex coordinates)**.** *Given two differentiable functions f and g of the momenta we have*

$$\{f, g\} = \sum_{l=1}^{n} \left(\frac{\partial f}{\partial \xi_l} \frac{\partial g}{\partial \eta_l} - \frac{\partial f}{\partial \eta_l} \frac{\partial g}{\partial \xi_l} \right) .$$

Proof. Cosnidering the case $n = 1$ for simplicity we find

$$\frac{\partial \xi}{\partial x} = \frac{1}{\sqrt{2}} = \frac{\partial \eta}{\partial y} , \qquad \frac{\partial \xi}{\partial y} = -\frac{i}{\sqrt{2}} = \frac{\partial \eta}{\partial x} .$$

So using the chain rule we get

$$\begin{aligned}
\{f, g\} &= \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \\
&= \frac{\partial f(\xi, \eta)}{\partial x} \frac{\partial g(\xi, \eta)}{\partial y} - \frac{\partial g(\xi, \eta)}{\partial x} \frac{\partial f(\xi, \eta)}{\partial y} \\
&= \left(\frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \left(\frac{\partial g}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial g}{\partial \eta} \frac{\partial \eta}{\partial y} \right) - \left(\frac{\partial g}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial g}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \left(\frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \\
&= \frac{\partial f}{\partial \xi} \frac{\partial g}{\partial \eta} - \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \xi} .
\end{aligned}$$

The general case is easily obtained reinstating the indices. □

Goal: construct a first integral of the interacting system H. We construct a formal power series starting as a perturbation of an I_l: the tentative first integral of the interacting system should take the form

$$\phi(x, y) = \phi_0(x, y) + \varepsilon \phi_1(x, y) + \varepsilon^2 \phi_2(x, y) + \varepsilon^3 \phi_3(x, y) + \ldots$$

where, for some arbitrarily picked l,

$$\phi_0(x, y) = I_l(x, y) ,$$

and ϕ_s is a homogeneous polynomial of degree $s + 2$.

Poincaré's construction We have to solve the Poisson bracket

$$\{\phi, H\} = 0$$

for ϕ, iteratively order by order in ε:

$$\{\phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \ldots , H_0 + \varepsilon H_1\} = 0 .$$

This yields the set of equations

$$\begin{aligned}
&\text{at order } \varepsilon^0 : & \{\phi_0, H_0\} &= 0 \\
&\text{at order } \varepsilon^1 : & \{\phi_1, H_0\} + \{\phi_0, H_1\} &= 0 \\
&\text{at order } \varepsilon^2 : & \{\phi_2, H_0\} + \{\phi_1, H_1\} &= 0 \\
&\text{at order } \varepsilon^3 : & \{\phi_3, H_0\} + \{\phi_2, H_1\} &= 0 .
\end{aligned}$$

Introducing the linear operator $\partial_{\boldsymbol{\omega}}$ (parametrized by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$ the frequencies appearing in H_0) acting on polynomials or functions by

$$\partial_{\boldsymbol{\omega}} f := \{f, H_0\}$$

we can write this system as

$$\partial_{\boldsymbol{\omega}} \phi_0 = 0 \ , \qquad \partial_{\boldsymbol{\omega}} \phi_s = \{H_1, \phi_{s-1}\} \quad \forall s \in \mathbb{N} \setminus \{0\} \ .$$

The first equation is satisfied by choosing $\phi_0 = I_l$ for some l. The second equation, called *homological equation*, shows us how to obtain iteratively the higher orders from the previously computed order (ϕ_{s-1} is known, solve for ϕ_s).

Note that in our case, the Hénon–Heiles model, all involved functions are polynomials, so that we can implement this construction as a symbolic computation on a computer with multi–variable polynomials.

Inversion of the Operator $\partial_{\boldsymbol{\omega}}$. Unfortunately the operator $\partial_{\boldsymbol{\omega}}$ has a non–trivial null space and is therefore not invertible (one can easily write down homogeneuous polynomials ψ such that $\partial_{\boldsymbol{\omega}} \psi = 0$). However, in some cases (such as the non–resonant Hénon–Heiles model), these do not appear, and on the relevant subspace the operator $\partial_{\boldsymbol{\omega}}$ can be inverted. Explicit inversion is achieved by diagonalizing it; this is achieved by using the complex coordinates as follows:

Lemma 1.2 (Diagonalization of $\partial_{\boldsymbol{\omega}}$). *The linear operator $\partial_{\boldsymbol{\omega}}$ is diagonal over the basis*

$$\xi^{\boldsymbol{j}} \eta^{\boldsymbol{k}} := \prod_{l=1}^{n} \xi_l^{j_l} \eta_l^{k_l} \ ,$$

where $\boldsymbol{j} = (j_1, j_2, \ldots) \in \mathbb{N}^n$ and $\boldsymbol{k} = (k_1, k_2, \ldots) \in \mathbb{N}^n$ are multi–indices. One has

$$\partial_{\boldsymbol{\omega}} \xi^{\mathbf{j}} \eta^{\mathbf{k}} = i \langle \boldsymbol{\omega}, \mathbf{j} - \mathbf{k} \rangle \xi^{\mathbf{j}} \eta^{\mathbf{k}}$$

with the Euclidean scalar product in $\mathbb{R}^n$ as

$$\langle \boldsymbol{\omega}, \mathbf{j} - \mathbf{k} \rangle := \sum_{m=1}^{n} \omega_m (j_m - k_m) \ .$$

Proof. Using lemma 1.1, check that for any differentable function f of the position and momenta we have

$$\partial_{\boldsymbol{\omega}} f = \sum_{m=1}^{n} i\omega_m \left(\xi_m \frac{\partial}{\partial \xi_m} - \eta_m \frac{\partial}{\partial \eta_m} \right) f \ .$$

One then computes

$$\begin{aligned} \partial_{\boldsymbol{\omega}} \xi_l^{j_l} \eta_l^{k_l} &= i\omega_l \xi_l \frac{\partial \xi_l^{j_l}}{\partial \xi_l} \eta_l^{k_l} - i\omega_l \eta_l \frac{\partial \eta_l^{k_l}}{\partial \eta_l} \xi_l^{j_l} \\ &= i\omega_l j_l \xi_l^{j_l} \eta_l^{k_l} - i\omega_l k_l \xi_l^{j_l} \eta_l^{k_l} \qquad = i\omega_l (j_l - k_l) \xi_l^{j_l} \eta_l^{k_l} \ . \end{aligned}$$

The multi–index case follows easily. □

Solving the Homological Equation We can therefore attempt to solve the equation $\partial_{\boldsymbol{\omega}}\phi = \psi$ for ϕ as follows.

- transform the polynomial ψ into complex coordinates, i. e., expand in the basis $\xi^{\mathbf{j}}\eta^{\mathbf{k}}$:
$$\psi = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}^n} \psi_{\mathbf{j},\mathbf{k}}\xi^{\mathbf{j}}\eta^{\mathbf{k}} \ .$$
- $\psi = \partial_{\boldsymbol{\omega}}\phi$ becomes, with $\phi_{\mathbf{j},\mathbf{k}}$ to be determined:
$$\sum_{\mathbf{j},\mathbf{k}} \psi_{\mathbf{j},\mathbf{k}}\xi^{\mathbf{j}}\eta^{\mathbf{k}} = \partial_{\boldsymbol{\omega}} \sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}^n} \phi_{\mathbf{j},\mathbf{k}}\xi^{\mathbf{j}}\eta^{\mathbf{k}} \tag{1.2}$$
$$= \sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}^n} \phi_{\mathbf{j},\mathbf{k}} i\langle\boldsymbol{\omega}, \mathbf{j}-\mathbf{k}\rangle\xi^{\mathbf{j}}\eta^{\mathbf{k}} \ . \tag{1.3}$$
- If all factors $i\langle\boldsymbol{\omega}, \mathbf{j}-\mathbf{k}\rangle \neq 0$ (we say that there are *no resonances*) then the equation is solved by setting
$$\phi_{\mathbf{j},\mathbf{k}} := \frac{-i}{\langle\boldsymbol{\omega}, \mathbf{j}-\mathbf{k}\rangle}\psi_{\mathbf{j},\mathbf{k}} \ .$$
In the following we are going to discuss the absence of resonances in the non–resonant Hénon–Heiles model, so that this procedure in fact works.

Absence of Resonances in non–resonant Hénon–Heiles. We say that the pair $(\mathbf{j}, \mathbf{k})$ is a *resonance* if

$$\langle\boldsymbol{\omega}, \mathbf{j}-\mathbf{k}\rangle = 0 \ .$$

Note that $\mathbf{j}-\mathbf{k} \in \mathbb{Z}^n$. Furthermore $n = 2$ in the Hénon–Heiles model. So if $\omega_1 \in \mathbb{Q}$ and $\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$, then the only solution $\mathbf{j}-\mathbf{k} \in \mathbb{Z}^n$ of $\langle\boldsymbol{\omega}, \mathbf{j}-\mathbf{k}\rangle = 0$ is

$$\mathbf{j}-\mathbf{k} = 0 \qquad \Leftrightarrow \qquad \mathbf{j} = \mathbf{k} \ .$$

One may verify that this does not happen in the non–resonant Hénon–Heiles model (theory to be discussed separately).

2 Implementation for non–resonant Hénon–Heiles

The goal is to implement the operations in terms of polynomials usign computer algebra. The relevant formulas are as follows:

$$H := H_0 + H_1 \qquad (\varepsilon = 1)$$
$$H_0 := \frac{\omega_1}{2}(y_1^2 + x_1^2) + \frac{\omega_2}{2}(y_2^2 + x_2^2) \ , \qquad \omega_1 := 1 \ , \ \omega_2 := \frac{\sqrt{5}-1}{2}$$
$$H_1 := x_1^2 x_2 - \frac{1}{3}x_2^3 \ .$$

We pick as the starting point of our construction

$$\phi_0 = I_1 \ .$$

With the complex coordinates as defined in eq. (1.1) the model takes the form

$$H_0 = \omega_1 I_1 + \omega_2 I_2 \; , \quad I_1 = i\xi_1\eta_1 \; , \quad I_2 = i\xi_2\eta_2$$
$$H_1 = \frac{1}{2^{2/3}}\Big(\xi_1^2\xi_2 + \xi_1^2 i\eta_2 - \eta_1^2\xi_2 - i\eta_1^2\eta_2 + 2i\xi_1\eta_1\xi_2 - 2\xi_1\eta_1\eta_2 - \frac{1}{3}\xi_2^3 - i\xi_2^2\eta_2 + \frac{1}{3}i\eta_2^3 + \xi_2\eta_2^2\Big)$$

For the computation by hand the following two lemmata are useful. (Instead, for the implementation on the computer it is easier to use lemma 1.1 to compute Poisson brackets!).

Lemma 2.1 (Poisson bracket expansion)**.** *For functions A, B, C we have*

$$\{AB, C\} = A\{B, C\} + \{A, C\}B \; .$$

Proof. trivial computation □

Lemma 2.2 (Fundamental Poisson brackets in complex coordinates)**.**

$$\{\xi, \xi\} = 0 = \{\eta, \eta\} \; , \qquad \{\eta, \xi\} = -1 \; , \qquad \{\xi, \eta\} = 1 \; .$$

Proof. trivial computation □

Repeatedly expanding the Poisson brackets until the fundamental ones can be used, one may also compute by hand that

$$\{H_1, i\xi_1\eta_1\} = \frac{1}{\sqrt{2}}\left(i\xi_1^2\xi_2 - \eta_2\xi_1^2 + i\xi_2\eta_1^2 - \eta_2\eta_1^2\right) \; . \tag{2.1}$$

Check that you get this right!

Solving the homological equation. We need to solve $\partial_{\boldsymbol{\omega}}\phi = \psi$ for ϕ, with the given $\psi = \{H_1, i\xi_1\eta_1\}$ that we just computed. The exponents are read of as follows for the four summands in eq. (2.1):

$$\begin{aligned}
&\xi_1^2\xi_2 = \xi_1^{j_1}\xi_2^{j_2}\eta_1^{k_1}\eta_2^{k_2} && \Rightarrow \quad \mathbf{j} = (2,1) \; , \quad \mathbf{k} = (0,0) \\
&\xi_1^2\eta_2 = \cdots && \Rightarrow \quad \mathbf{j} = (2,0) \; , \quad \mathbf{k} = (0,1) \\
&\xi_2\eta_1^2 && \Rightarrow \quad \mathbf{j} = (0,1) \; , \quad \mathbf{k} = (2,0) \\
&\eta_2\eta_1^2 && \Rightarrow \quad \mathbf{j} = (0,0) \; , \quad \mathbf{k} = (2,1) \; .
\end{aligned}$$

Note that the resonant case $\mathbf{j} = \mathbf{k}$ never appears, so there is no problem.

We make the ansatz

$$\phi = a\xi_1^2\xi_2 + b\eta_2\xi_1^2 + c\xi_2\eta_1^2 + d\eta_2\eta_1^2 \; ,$$

with $a, b, c, d \in \mathbb{C}$ to be determined, and plug it into the homological equation. On the left hand side we find

$$\partial_{\boldsymbol{\omega}}\phi = i\langle\boldsymbol{\omega}, \begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\rangle \; a \; \xi_1^2\xi_2 + \ldots$$

and on the right hand side

$$\psi = \frac{i}{\sqrt{2}} \qquad\qquad\qquad \xi_1^2\xi_2 + \dots$$

Comparing term by term we get

$$a = \frac{1}{\sqrt{2}} \frac{1}{\langle \boldsymbol{\omega}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle}, \qquad b = \frac{i}{\sqrt{2}} \frac{1}{\langle \boldsymbol{\omega}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rangle}, \qquad \dots$$

The total first order correction is then

$$\begin{aligned} \phi_1 &= \frac{1}{\sqrt{2}} \Bigg(\langle \boldsymbol{\omega}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle^{-1} \xi_1^2\xi_2 + i \langle \boldsymbol{\omega}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rangle^{-1} \eta_2\xi_1^2 \\ &\qquad + \langle \boldsymbol{\omega}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \rangle^{-1} \xi_2\eta_1^2 + i \langle \boldsymbol{\omega}, \begin{pmatrix} -2 \\ -1 \end{pmatrix} \rangle^{-1} \eta_2\eta_1^2 \Bigg) \\ &= \frac{1}{\sqrt{2}} \Bigg(\frac{1}{2\omega_1 + \omega_2} \xi_1^2\xi_2 + i \frac{1}{2\omega_1 - \omega_2} \eta_2\xi_1^2 \\ &\qquad + \frac{1}{-2\omega_1 + \omega_2} \xi_2\eta_1^2 + i \frac{1}{-2\omega_1 - \omega_2} \eta_2\eta_1^2 \Bigg) . \end{aligned}$$

Again, check that you get this right both by hand and in your program.

Solution Your program should produce the following result for the first few orders (don't forget the prefactor $1/\sqrt{2}$ to get the numerical coefficients right):

$$\begin{aligned} \phi_1 &= (0.27)\, \xi_1^2\xi_2 + (0.51i)\, \xi_1^2\eta_2 + (-0.51)\, \eta_1^2\xi_2 + (-0.27i)\, \eta_1^2\eta_2 \\ \phi_2 &= (0.02)\, \xi_1^4 + (0.09i)\, \xi_1^3\eta_1 + (-0.14)\, \xi_1^2\xi_2^2 + (-0.64i)\, \xi_1^2\xi_2\eta_2 + (1.06)\, \xi_1^2\eta_2^2 \\ &\quad + (-0.09i)\, \xi_1\eta_1^3 + (0.28i)\, \xi_1\eta_1\xi_2^2 + (-0.28i)\, \xi_1\eta_1\eta_2^2 + (0.02)\, \eta_1^4 + (1.06)\, \eta_1^2\xi_2^2 \\ &\quad + (0.64i)\, \eta_1^2\xi_2\eta_2 + (-0.14)\, \eta_1^2\eta_2^2 \\ \phi_3 &= (-0.03)\xi_1^4\xi_2 + (-0.15i)\xi_1^4\eta_2 + (-0.24i)\xi_1^3\eta_1\xi_2 + (0.75)\xi_1^3\eta_1\eta_2 + (-0.01)\xi_1^2\eta_1^2\xi_2 \\ &\quad + (-0.01i)\xi_1^2\eta_1^2\eta_2 + (0.14)\xi_1^2\xi_2^3 + (0.79i)\xi_1^2\xi_2^2\eta_2 + (-2.72)\xi_1^2\xi_2\eta_2^2 + (-15.19i)\xi_1^2\eta_2^3 \\ &\quad + (0.75i)\xi_1\eta_1^3\xi_2 + (-0.24)\xi_1\eta_1^3\eta_2 + (-0.59i)\xi_1\eta_1\xi_2^3 + (1.78)\xi_1\eta_1\xi_2^2\eta_2 \\ &\quad + (1.78i)\xi_1\eta_1\xi_2\eta_2^2 + (-0.59)\xi_1\eta_1\eta_2^3 + (-0.15)\eta_1^4\xi_2 + (-0.03i)\eta_1^4\eta_2 + (-15.19)\eta_1^2\xi_2^3 \\ &\quad + (-2.72i)\eta_1^2\xi_2^2\eta_2 + (0.79)\eta_1^2\xi_2\eta_2^2 + (0.14i)\eta_1^2\eta_2^3 \end{aligned}$$

Check that you get at least ϕ_2 right here!

This is an early draft of lectures notes for the computer lab course of "Hamiltonian Systems 1", work in progress.