Effective Dynamics of Interacting Fermions

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Many–Body Schrödinger Equation

Quantum System of N Fermions

Hamilton operator of N identical spinless particles:

$$H_N := \sum_{i=1}^{N} (-\Delta_i) + \lambda \sum_{1 \leq i < j \leq N} V(x_i - x_j) \qquad \text{with } V : \mathbb{R}^3 \to \mathbb{R} \; .$$

Acts on the L^2–subspace of antisymmetric wave functions of $3N$ variables

$$\psi(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}) = \operatorname{sgn}(\sigma)\psi(x_1, x_2, \ldots, x_N) \qquad \forall \sigma \in S_N \; .$$

For reasonable potentials, the Hamiltonian is self–adjoint (e. g., Kato–Rellich theorem).

Time evolution is described by Schrödinger equation:

$$\left.\begin{array}{r} i\partial_t \psi_t = H_N \psi_t \\ \text{initial data } \psi_0 \end{array}\right\} \quad \Leftrightarrow \quad \psi_t = e^{-iH_N t}\psi_0 \; .$$

Explicit Solutions?

- Analytical solutions up to $N = 2$ (in center–of–mass coordinates), or $N = 3$ (some examples with high symmetry)
- Numerical methods (quantum Monte Carlo) are limited by exponential growth of Hilbert space dimension: "curse of dimension"

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We need approximations!

- There is no one-size-fits-all approximation!
  Range of phenomena described by the Schrödinger equation is far too large: superconductors, neutron stars, electric vehicles,. . .
  - Specify particular physical situations — mathematical idealization: scaling limits.
  - Specify quantities to be approximated: which observables, which excitations, . . . ?

Fermionic Mean–Field Scaling

Mean–Field Regime = High Density & Weak Interaction

- Gas at high density with weak interaction.
  In the limit, every particle moves in a continuous cloud generated by all the other particles, "moves in mean field".

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- **High density:** N fermions (at least initially) in external trapping potential or fixed–size torus and $N \to +\infty$
- **"Weak" interaction?** Minimize $\langle \psi, \sum_{j=1}^{N} (-\Delta_j) \psi \rangle$! Antisymm. tensor product

$$\psi = \frac{1}{N!} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma)\, \varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(N)}$$

of eigenfunctions of the Laplacian $\varphi_j(x) := (2\pi)^{-3/2} e^{i k_j \cdot x}$, $k_j \in \mathbb{Z}^3$:

$$\sum_{j=1}^{N} |k_j|^2 = \sum_{|k| \leq c N^{1/3}} |k|^2 \sim N^{5/3} \qquad \text{c. f.} \qquad \langle \psi, \lambda \sum_{1 \leq i < j \leq N} V(x_i - x_j) \psi \rangle \sim \lambda N^2 \,.$$

fermionic mean–field scaling: $\lambda = N^{-1/3}$ (bosons: $\lambda = N^{-1}$)

Semiclassical Time Scale

- Velocity $\sim$ highest momenta $k \sim N^{1/3}$.
  A particle traverses the entire torus in a time of order $N^{-1/3}$.
  No significant loss in considering only times $t = N^{-1/3}\tau$, where $\tau \sim 1$:

$$iN^{1/3}\partial_\tau \psi_\tau = \left[\sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N^{1/3}} \sum_{1 \le i < j \le N} V(x_i - x_j)\right] \psi_\tau \, .$$

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- Trivial step: define effective Planck constant $\hbar := N^{-1/3}$ and multiply by $\hbar^2$

Mean-field scaling is naturally coupled to a semiclassical scaling:

$$i\hbar\partial_\tau \psi_\tau = \left[\sum_{j=1}^{N} -\hbar^2\Delta_{x_j} + \frac{1}{N} \sum_{1\le i<j\le N} V(x_i - x_j)\right]\psi_\tau \qquad \text{with } \hbar = N^{-1/3}\,.$$

Goal: Approximate ψ_τ by simpler initial value problems.

Effective Theories

- **Vlasov equation:**
  theory on classical phase space, no quantum effects retained, "semiclassical"
- **Hartree–Fock equation:**
  quantum, only the unavoidable entanglement due to antisymmetry of fermionic wave functions (kinematic entanglement)
- **Random Phase Approximation:**
  quantum, entanglement of particle–hole pairs (dynamical entanglement, to leading order)

Caution: $\{\text{Vlasov}, \text{HF}, \text{RPA}\}$ is not an ordered set (not transitive, not antisymmetric):

- For practical purposes simpler equations sometimes work better!
- Do we enlarge or restrict the set of permitted initial data?
- More effects neglected — more mathematical work to estimate them?

Vlasov Equation

Classical Approximation

- In classical mechanics a system is described by a particle density on phase phase:

$$f : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty) \;, \qquad \int f(x,p) \mathrm{d}x \mathrm{d}p = 1 \;.$$

- Classical mean–field evolution for f_τ: Vlasov equation

$$\frac{\partial f_\tau}{\partial \tau} + 2p \cdot \nabla_x f_\tau = -F(f_\tau) \cdot \nabla_p f_\tau$$

where

$$F(f_\tau) := -\nabla(V * \rho_{f_\tau}) \;, \quad \rho_{f_\tau}(x) := \int f_\tau(x,p) \mathrm{d}p \;.$$

From Quantum to Classical

- From quantum mechanics to phase space: For $\psi \in L^2(\mathbb{R}^3)^{\otimes N}$, define the one–particle reduced density matrix

$$\gamma_\psi := N \operatorname{tr}_{2,\dots,N} |\psi\rangle\langle\psi|$$

and then the Wigner transform

$$W_\psi(x,p) := \frac{1}{(2\pi)^3} \int e^{-ip\cdot y/\hbar} \, \gamma_\psi\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \mathrm{d}y \, .$$

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- *Narnhofer–Sewell* '81: W_{ψ_τ} converges to solution of Vlasov equation for analytic V.
- *Spohn* '81: Generalization to twice differentiable V.
- Recent results, in particular concerning singular V such as Coulomb potential: Saffirio, Thursday 11:30

Hartree–Fock Approximation

Hartree-Fock Approximation

Restrict QM to antisymmetrized tensor products $\psi = \mathcal{A}(\varphi_1 \otimes \ldots \otimes \varphi_N)$ (no other linear combinations permitted) and optimize the choice of the $\varphi_j \in L^2(\mathbb{R}^3)$.

- Approximate time evolution

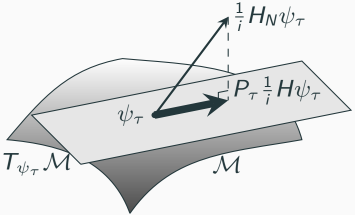
$$e^{-iH_N\tau/\hbar}\mathcal{A}(\varphi_{1,0} \otimes \ldots \otimes \varphi_{N,0}) \simeq \mathcal{A}(\varphi_{1,\tau} \otimes \ldots \otimes \varphi_{N,\tau})$$

- Hartree-Fock equations, for $i = 1, 2, \ldots N$:

$$i\hbar\partial_\tau\varphi_{i,\tau} = -\hbar^2\Delta\varphi_{i,\tau} + \frac{1}{N}\sum_{j=1}^{N}\left(V * |\varphi_{j,\tau}|^2\right)\varphi_{i,\tau} - \frac{1}{N}\sum_{j=1}^{N}\left(V * (\varphi_{i,\tau}\overline{\varphi_{j,\tau}})\right)\varphi_{j,\tau}$$

Dirac–Frenkel principle:

Submanifold $\mathcal{M} \subset \mathcal{H}$



P_τ = orthog. projection on $T_{\psi_\tau}\mathcal{M}$

[Lubich '08, B–Sok–Solovej '18]

Rigorous Error Estimates

- *Erdős–Elgart–Schlein–Yau* '04: Convergence from Schrödinger equation to Hartree–Fock equation for short times, $\tau < \tau_0$. Analytic V.
- Hartree–Fock equation for scalings with weaker interaction or shorter time scale:
  - *Bardos–Golse–Gottlieb–Mauser* '03
  - *Fröhlich–Knowles* '11
  - *Pickl–Petrat '14*
  - *Bach–Breteaux–Petrat–Pickl–Tzaneteas '16*.
- *B–Porta–Schlein* '14: $V \in L^1(\mathbb{R}^3)$ with $\int |\hat{V}(p)|(1+|p|)^2 \mathrm{d}p < \infty$, arbitrary τ.
- generalizations: mixed states *B–Jakšić–Porta–Saffirio–Schlein* '16, singular interactions: *Chong, Lafleche, Leopold, Saffirio*

One–Particle Density Matrix

- For $\psi \in L^2(\mathbb{R}^3)^{\otimes N}$, the one–particle density matrix is (as before)

$$\gamma_\psi := N \,\mathrm{tr}_{2,\dots,N} |\psi\rangle\langle\psi| \,.$$

- If ψ is an antisymmetrized tensor product, γ_ψ is a projection in $L^2(\mathbb{R}^3)$:

$$\psi = \mathcal{A}(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_N) \quad \Leftrightarrow \quad \gamma_\psi = \sum_{j=1}^{N} |\varphi_j\rangle\langle\varphi_j| \,.$$

- Hartree–Fock equations:

$$i\hbar\partial_t \gamma_t^{\mathrm{HF}} = \left[-\hbar^2 \Delta + V * \rho_t - X_t \,,\; \gamma_t^{\mathrm{HF}} \right] ,$$

with the multiplication operator $V * \rho_t(x) = N^{-1} \int V(x-y)\gamma_t^{\mathrm{HF}}(y;y)\mathrm{d}y$, and X_t the operator with integral kernel $N^{-1} V(x-y)\gamma_t^{\mathrm{HF}}(x;y)$.

Theorem (B–Porta–Schlein '14)

Let $V \in L^1(\mathbb{R}^3)$ *with* $\int |\hat{V}(p)|(1+|p|)^2 dp < \infty$.

Let $\{\varphi_j\}_{j=1}^{\infty}$ *be an orthonormal basis in* $L^2(\mathbb{R}^3)$.
Let $\psi_0 = \mathcal{A}(\varphi_1 \otimes \ldots \otimes \varphi_N)$. *Assume semiclassical commutators bounds*

$$\|[x_i, \gamma_{\psi_0}]\|_{\text{tr}} \leq CN\hbar\,, \qquad \|i\hbar\partial_i, \gamma_{\psi_0}]\|_{\text{tr}} \leq CN\hbar\,.$$

Let

- γ_{ψ_t}: *one–particle reduced density matrix of the solution of the Schrödinger equation with initial data* ψ_0,
- γ_t^{HF}: *solution of HF equation with initial data* γ_{ψ_0}.

Then

$$\|\gamma_{\psi_t} - \gamma_t^{HF}\|_{\text{tr}} \leq CN^{1/6} e^{ce^{c|t|}} \qquad (\text{compare } \operatorname{tr}\gamma_{\psi_t} = N = \operatorname{tr}\gamma_t^{HF})\,.$$

Construction of Initial Data

We require an $\hbar$–gain in commutators with position and momentum:

$$\|[x_i, \gamma_{\psi_0}]\|_{\mathrm{tr}} \leq CN\hbar\,, \qquad \|i\hbar\partial_i, \gamma_{\psi_0}]\|_{\mathrm{tr}} \leq CN\hbar\,.$$

Verified for non–interacting fermions in different situations:

- translation invariant state: plane waves on torus (but that is stationary under the HF evolution even when the interaction is switched on)
- in general trapping potentials [Fournais–Mikkelsen '19]: by semiclassical analysis
- in an (anisotropic) harmonic trap: by explicit computation

Experimentally: quantum quench, prepare non–interacting fermions in ground state, than switch on the interaction by a Feshbach resonance.

Proof of the [BPS14] Theorem

Second Quantization

- Fermionic Fock space

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{A}\, L^2(\mathbb{R}^{3n})\,, \qquad \psi = (\psi^{(0)}, \psi^{(1)}, \dots, \psi^{(N)}, \dots) \in \mathcal{F}$$

- Canonical anticommutation relations

$$\{a_x, a_y^*\} = \delta(x-y)\,, \quad \{a_x, a_y\} = \{a_y^*, a_x^*\} = 0.$$

- On $(0, \dots, 0, \psi^{(N)}, 0, \dots) \in \mathcal{F}$ we have $\mathcal{H} = H_N$ by defining

$$\mathcal{H} := \hbar^2 \int \mathrm{d}x\, \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N} \int \mathrm{d}x \mathrm{d}y\, V(x-y) a_x^* a_y^* a_y a_x$$

- Vacuum $\Omega = (1, 0, 0, 0, \dots) \in \mathcal{F}$
- Number operator

$$\mathcal{N} = \int a_x^* a_x \,\mathrm{d}x$$

Particle–Hole Transformation (remember for the RPA section!)

Use a unitary $R : \mathcal{F} \to \mathcal{F}$ to represent Fock space as excitations (particles or holes) over the Hartree–Fock state, instead of particles over vacuum.

$$R\Omega := \mathcal{A}(\varphi_1 \otimes \ldots \otimes \varphi_N) \in \mathcal{F}$$

$$Ra^*(\varphi_i)R^* := \begin{cases} a^*(\varphi_i) & \text{for } i > N \quad \text{(creates particle)} \\ a(\varphi_i) & \text{for } i \leq N \quad \text{(creates hole).} \end{cases}$$

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- This is a Bogoliubov transformation:

$$Ra^*_x R^* = a^*(u_x) + a(v_x),$$

with $v = \sum_{j=1}^{N} |\varphi_j\rangle\langle\varphi_j|$, $u = \mathbb{1} - v$ (up to conjugations), and $v_x(y) := v(y, x)$.

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- Analogously, for $\varphi_{j,\tau}$ solving the HF equations, introduce R_τ such that

$$R_\tau\Omega = \mathcal{A}(\varphi_{1,\tau} \otimes \ldots \otimes \varphi_{N,\tau}) \, .$$

$\|\gamma_{\psi_\tau} - \gamma_\tau^{\mathbf{HF}}\|_{\mathrm{tr}} \leq$ **Number of Excitations**

- Number of excitations w. r. t. the HF-evolved state:

$$\mathcal{N}_\tau^{\mathrm{exc}} := R_\tau \mathcal{N} R_\tau^* .$$

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- A short calculation shows

$$\begin{aligned} \|\gamma_{\psi_\tau} - \gamma_\tau^{\mathrm{HF}}\|_{\mathrm{tr}} &\leq C N^{1/2} \langle e^{-i\mathcal{H}\tau/\hbar} R_0 \Omega, \mathcal{N}_\tau^{\mathrm{exc}} e^{-i\mathcal{H}\tau/\hbar} R_0 \Omega \rangle \\ &= C N^{1/2} \langle U(\tau)\Omega, \mathcal{N} U(\tau)\Omega \rangle \end{aligned}$$

with $U(\tau) := R_\tau^* e^{-i\mathcal{H}\tau/\hbar} R_0$.

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  with $U(\tau) := R_\tau^* e^{-i\mathcal{H}\tau/\hbar} R_0$.

- To control the trace norm difference, it is enough to show that

$$\langle U(\tau)\Omega, \mathcal{N} U(\tau)\Omega\rangle = \mathcal{O}(1)\ .$$

- By Grönwall's lemma, it is sufficient to prove

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle U(\tau)\Omega, \mathcal{N} U(\tau)\Omega\rangle \leq C_t \langle U(\tau)\Omega, \mathcal{N} U(\tau)\Omega\rangle.$$

Cancellations

- With the generator defined by $i\hbar\partial_\tau U(\tau) = \mathcal{L}_N(\tau)U(\tau)$ we have to show

$$|i\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}\langle U(\tau)\Omega, \mathcal{N} U(\tau)\Omega\rangle| = |\langle U(\tau)\Omega, [\mathcal{L}_N(\tau), \mathcal{N}]U(\tau)\Omega\rangle| \leq \hbar C_\tau \langle U(\tau)\Omega, \mathcal{N} U(\tau)\Omega\rangle.$$

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  using the HF equation the biggest terms of $\mathcal{L}_N(\tau)$ cancel!

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- $U(\tau)$ depends on R^*_τ which depends on the HF equation;
  using the HF equation the biggest terms of $\mathcal{L}_N(\tau)$ cancel!
- Remaining:

$$\begin{aligned}&\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}\langle U(\tau)\Omega, \mathcal{N}U(\tau)\Omega\rangle\\ &\simeq \frac{1}{N}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\langle U(\tau)\Omega,\, a^*(u_{\tau,y})a(u_{\tau,y})a(v_{\tau,x})a(u_{\tau,x})U(\tau)\Omega\rangle\ .\end{aligned}$$

- Easy bound $\mathcal{O}(\mathcal{N})$, but need $\mathcal{O}(\hbar\mathcal{N})$.

Using the Semiclassical Commutators

- Have to extract a factor $\hbar$:

$$\frac{1}{N}\int \mathrm{d}x\mathrm{d}y\, V(x-y)\langle U(\tau)\Omega, a^*(u_{\tau,y})a(u_{\tau,y})a(v_{\tau,x})a(u_{\tau,x})U(\tau)\Omega\rangle.$$

Recall: $v = v^2$, $u = \mathbb{1} - v$:

$$\int \mathrm{d}x\, v_{\tau,x}u_{\tau,x} = 0.$$

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- The variables x and y can be treated separately using the Fourier decomposition $V(x-y) = \sum_{p\in\mathbb{Z}^3} \hat{V}(p)\, e^{ip\cdot x}e^{-ip\cdot y}$:

$$\int \mathrm{d}x\, v_{\tau,x}e^{ip\cdot x}u_{\tau,x} = \int \mathrm{d}x\, v_{\tau,x}[e^{ip\cdot x}, u_\tau](\cdot,x) = \int \mathrm{d}x\, v_{\tau,x}\underbrace{[e^{ip\cdot x}, \gamma_\tau^{\mathrm{HF}}](\cdot,x)}_{\simeq CN\hbar}.$$

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Random Phase Approximation

Back to the Particle–Hole Transformation

Our approach to RPA: start from the Fermi ball of the Hamiltonian on the torus.
The Fermi ball is stationary under HF evolution. Consider its excitations.

In momentum representation the particle–hole transformation acts as

$$R\, a_k^* \, R^* := \begin{cases} a_k^* & |k| > (\frac{3}{4\pi})^{1/3} N^{1/3} \\ a_k & |k| \leq (\frac{3}{4\pi})^{1/3} N^{1/3} \,. \end{cases}$$

Expand $R^* H_N R$ and normal–order

$$R^* H_N R = \underbrace{E_N^{\text{pw}} + \hbar^2 \sum_{p \in \mathcal{B}_F^c} p^2 a_p^* a_p - \hbar^2 \sum_{h \in \mathcal{B}_F} h^2 a_h^* a_h}_{=: \; H^{\text{kin}}} \; + \underbrace{X}_{\substack{\text{exchange term,} \\ \text{negligible}}} + \underbrace{Q}_{\substack{\text{quartic in} \\ \text{operators } a^* \text{ and } a}}$$

Goal: a quadratic approximation to the excitation Hamiltonian $H^{\text{kin}} + Q$.

(Quadratic Hamiltonians can be diagonalized by Bogoliubov transformations.)

Bosonization of the Interaction

Observe: if we introduce collective pair operators

$$b_k^* := \sum_{\substack{p \in \mathcal{B}_F^c \\ h \in \mathcal{B}_F}} \delta_{p-h,k} a_p^* a_h^*$$

p "particle" outside the Fermi ball

h "hole" inside the Fermi ball

then

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \Big(2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \Big) + \mathcal{O}\Big(\frac{\mathcal{N}^2}{N} \Big) .$$

This is convenient because the b_k^* and b_k have approximately bosonic commutators:

$$[b_k^*, b_l^*] = 0 \quad , \qquad [b_l, b_k^*] = \delta_{k,l} n_k^2 + \mathcal{E}(k,l) .$$

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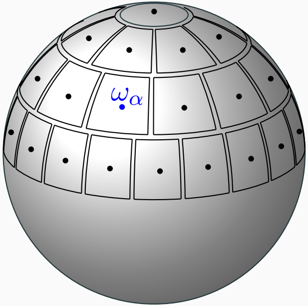
But how to express H^{kin} through pair operators?

Bosonization of the Kinetic Energy

Fermi ball $\mathcal{B}_F$



[Benfatto–Gallavotti '90]
[Haldane '94]
[Fröhlich–Götschmann–Marchetti '95]
[Kopietz et al. '95]

Localize to $M = M(N)$ patches near the Fermi surface,

$$b^*_{\alpha,k} := \frac{1}{n_{\alpha,k}} \sum_{\substack{p \in \mathcal{B}_F^c \cap B_\alpha \\ h \in \mathcal{B}_F \cap B_\alpha}} \delta_{p-h,k} a^*_p a^*_h$$

with $n_{\alpha,k}$ chosen to normalize $\|b^*_{\alpha,k}\Omega\| = 1$.

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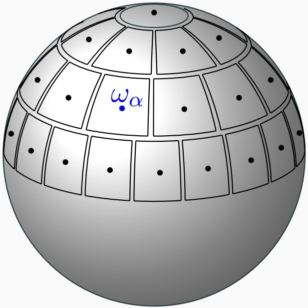

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Linearize kinetic energy around patch center ω_α:

$$[H^{\text{kin}}, b^*_{\alpha,k}] \simeq 2\hbar|k \cdot \hat{\omega}_\alpha| b^*_{\alpha,k}$$

We approximate

$$H^{\text{kin}} \simeq \sum_{k \in \mathbb{Z}^3} \sum_{\alpha=1}^{M} 2\hbar u_\alpha(k)^2 b^*_{\alpha,k} b_{\alpha,k}\,, \quad u_\alpha(k)^2 := |k \cdot \hat{\omega}_\alpha|\,.$$

Decomposing the Interaction over Patches

Recall

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \left(2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k\right)$$

Decompose

$$b_k^* = \sum_{\alpha=1}^{M} n_{\alpha,k} b_{\alpha,k}^* + \text{lower order} \; .$$

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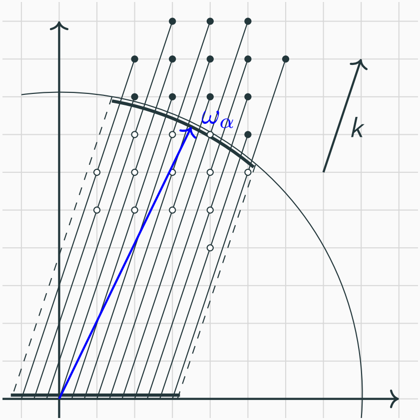
Normalization:

$$n_{\alpha,k}^2 = \#\text{p-h pairs in patch } B_\alpha \text{ with momentum } k$$

$$\simeq \frac{4\pi N^{2/3}}{M} |k \cdot \hat{\omega}_\alpha| = \frac{4\pi N^{2/3}}{M} u_\alpha(k)^2 .$$



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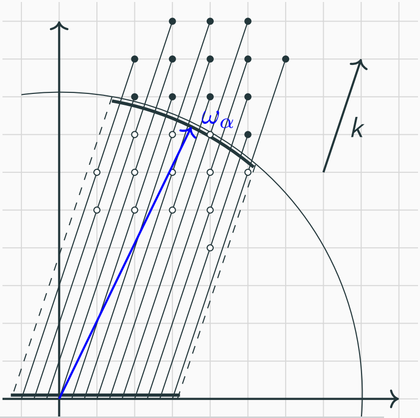

Effective Quadratic Bosonic Hamiltonian

$$H^{\text{eff}} = \hbar \sum_{k \in \mathbb{Z}^3} \left[\sum_\alpha u_\alpha(k)^2 b_{\alpha,k}^* b_{\alpha,k} + \frac{\hat{V}(k)}{M} \sum_{\alpha,\beta} \left(u_\alpha(k) u_\beta(k) b_{\alpha,k}^* b_{\beta,k} + u_\alpha(k) u_\beta(k) b_{\alpha,k}^* b_{\beta,-k}^* + \text{h.c.} \right) \right]$$

Bogoliubov Diagonalization

Quadratic Hamiltonians can be diagonalized by a Bogoliubov transformation

$$T = \exp\left(\sum_{k \in \mathbb{Z}^3} \sum_{\alpha,\beta=1}^{M} K(k)_{\alpha,\beta} b^*_{\alpha,k} b^*_{\beta,-k} - \text{h.c.} \right) .$$

Expanding into commutators we find

$$T^* b_{\alpha,k} T \simeq \sum_{\beta=1}^{M} \cosh(K(k))_{\alpha,\beta} b_{\beta,k} + \sum_{\beta=1}^{M} \sinh(K(k))_{\alpha,\beta} b^*_{\beta,-k}$$

and choose the $M \times M$–matrix $K(k)$ to make $b^* b^*$– and bb–terms vanish from H^{eff}:

$$T^* H^{\text{eff}} T \simeq E_N^{\text{RPA}} + \hbar \sum_{k \in \mathbb{Z}^3} \sum_{\alpha,\beta=1}^{M} E(k)_{\alpha,\beta} b^*_{\alpha,k} b_{\beta,k} .$$

In particular, the ground state of H^{eff} is $\xi_{\text{gs}} \simeq T\Omega$, and therefore the ground state of H_N is approximately $RT\Omega$. We add bosonic excitations and follow their evolution!

Effective Bosonic Evolution

Note that this is an (approximately) bosonic second quantization:

$$T^* H^{\text{eff}} T \simeq E_N^{\text{RPA}} + \hbar \sum_{k \in \mathbb{Z}^3} \sum_{\alpha,\beta=1}^{M} E(k)_{\alpha,\beta} b^*_{\alpha,k} b_{\beta,k}$$

$$\simeq E_N^{\text{RPA}} + \mathrm{d}\Gamma_{\text{bosonic}} \underbrace{\Big(\hbar \bigoplus_{k \in \mathbb{Z}^3} E(k) \Big)}_{=:\ H_{\mathrm{B}}} .$$

Consider a one–boson wave function

$$\eta \in \mathfrak{h}_{\mathrm{B}} := \bigoplus_{k \in \mathbb{Z}^3} \mathbb{C}^M .$$

Then

$$\eta_t := e^{-iH_B \tau/\hbar} \eta_0$$

is the time–evolution in the (first quantized) one–boson space.

For $\eta \in \mathfrak{h}_{\mathrm{B}}$ let

$$b^*(\eta) := \sum_{k \in \mathbb{Z}^3} \sum_{\alpha=1}^{M} b^*_{\alpha,k} \eta(k)_\alpha \,.$$

Theorem (B–Nam–Porta–Schlein–Seiringer '21)

Assume that $\hat{V}(p)$ is compactly supported and non–negative. Let

$$\xi_0 := \frac{1}{Z_m} b^*(\eta_1) \cdots b^*(\eta_m)\Omega \,, \qquad \xi_t := \frac{1}{Z_m} b^*(\eta_{1,\tau}) \cdots b^*(\eta_{m,\tau})\Omega \,.$$

Then

$$\| e^{-iH_N \tau/\hbar} RT\xi_0 - e^{-i(E_N^{\mathrm{pw}} + E_N^{\mathrm{RPA}})\tau/\hbar} RT\xi_\tau \| \leq C_{m,V} \hbar^{1/15} |\tau| \,.$$

If $H_{\mathrm{B}}\eta_i = e_i \eta_i$ ($e_i \in \mathbb{R}$) then we have constructed an approximate eigenstate of the many–body Hamiltonian, evolving up to times $|\tau| \ll N^{1/45}$ just with a phase:

$$e^{-iH_N\tau/\hbar} RT\xi_0 \simeq e^{-i\left(E_N^{\mathrm{pw}} + E_N^{\mathrm{RPA}} + \sum_{j=1}^m e_j\right)\tau/\hbar} RT\xi_0 \,.$$

Thank you!