Problem (1)

Given a Hilbert space $\mathcal{H}$ and a strongly continuous unitary group, $U(t)$:
(SCUG)

$$U(t)\psi = e^{i\alpha(t)}\psi \quad \text{for } \psi \in \mathcal{H}, \psi \neq 0$$

and $\alpha : \mathbb{R} \to \mathbb{R}$ continuous with $\alpha(0) = 0$

To prove.

a) $\alpha(t+s) = \alpha(t) + \alpha(s) \quad \forall \ s, t \in \mathbb{R}$

$U(t)$ is (SCUG)

then

$$U(t+s)\psi = e^{i\alpha(t+s)}\psi$$

from the composition property of $U(t)$, we know that

$$U(t+s) = U(t)U(s)$$

then

$$U(t)U(s)\psi = e^{i\alpha(t)}e^{i\alpha(s)}\psi$$

$$\Leftrightarrow \quad e^{i\alpha(t+s)}\psi = e^{i(\alpha(t)+\alpha(s))}\psi$$

$$\alpha(t+s) + 2k\pi = \alpha(t) + \alpha(s)$$

we know $\alpha(0) = 0$

then for $t = 0 = s$

$$\alpha(0) + 2k\pi = 0 \quad \Rightarrow k = 0$$

Hence $\alpha(t+s) = \alpha(t) + \alpha(s)$

b) <u>To prove</u> $\alpha(t) = \alpha(1)\,t \quad \forall t \in \mathbb{R}$

- Let $t = 0$

$$\alpha(0) = 0\,\alpha(1)$$

- let $t \in \mathbb{N}$

$$\alpha(t) = \alpha(\underbrace{1+1+\cdots+1}_{t\text{-times}}) \overset{(a)}{=} \alpha(1)+\alpha(1)+\cdots+\alpha(1)$$

$$= t\,\alpha(1)$$

- Let $t \in \mathbb{Q}$ s.t $t = p/q \quad \forall p, q \in \mathbb{N}$

$$\alpha(t) = \alpha(p/q) = p\,\alpha\left(\frac{1}{q}\right) = p\left(\frac{1}{q}(\alpha(1))\right) = t\,\alpha(1)$$

- Let $t \in \mathbb{R}$

$\exists \{t_k\}_{k\in\mathbb{N}} \subseteq \mathbb{Q}$ s.t. $\lim\limits_{k\to\infty} t_k \longrightarrow t$

as $\mathbb{Q}$ is dense in $\mathbb{R}$

then $\alpha(\lim\limits_{k\to\infty} t_k) \underset{\alpha\text{ continuous}}{=} \lim\limits_{k\to\infty} \alpha(t_k) = \lim\limits_{k\to\infty} t_k\,\alpha(1)$

$$= \alpha(1)\,t$$

Problem 2

We start with [II]

$$\mathcal{U}_{A_1, v_1, b_1, s_1} \mathcal{U}_{A_2, v_2, b_2, s_2} = e^{i\omega(\cdot,\cdot)} \mathcal{U}_{A_1 A_2,\, v_1 + A_1 v_2,\, b_1 + v_1 s_2 + A_1 b_2,\, s_1 + s_2}$$

LHS =

$$= U(-s_1)\, \mathcal{U}_{A_1, v_1, b_1, 0}\; U(-s_2)\, \mathcal{U}_{A_2, v_2, b_2, 0}$$

$$\overset{[VIII]}{=} e^{i\omega_2(A_1, v_1, b_1, -s_1)} \mathcal{U}_{A_1, v_1, b_1 - v_1 s_1, 0}\; U(-s_1)\, U(-s_2)\, \mathcal{U}_{A_2, v_2, b_2, 0}$$

$$= e^{i\omega_2(A_1, v_1, b_1, -s_1)} \mathcal{U}_{A_1, v_1, b_1 - v_1 s_1, 0}\; U(-s_1 - s_2)\, \mathcal{U}_{A_2, v_2, b_2, 0}$$

$$\overset{[VIII]}{=} e^{i\omega_2(A_1, v_1, b_1, -s_1)} \mathcal{U}_{A_1, v_1, b_1 - v_1 s_1, 0}\; e^{i\omega_2(A_2, v_2, b_2, -s_1 - s_2)}$$

$$\cdot\, \mathcal{U}_{A_2, v_2, b_2 + v_2(-s_1 - s_2), 0}\; U(-s_1 - s_2)$$

$$= e^{i\omega_2(A_1, v_1, b_1, -s_1) + i\omega_2(A_2, v_2, b_2, -s_1 - s_2)} \mathcal{U}_{A_1, v_1, b_1 - v_1 s_1, 0}\, \cdot$$

$$\cdot\, \mathcal{U}_{A_2, v_2, b_2 - v_2(s_1 + s_2), 0}\; U(-s_1, -s_2)$$

$$= e^{i\omega_2(A_1, v_1, b_1, -s_1) + i\omega_2(A_2, v_2, b_2, -s_1 - s_2)}\, \cdot$$

$$\cdot\, e^{i\omega_2(A_1, v_1, b_1 - v_1 s_1, 0,\, A_2, v_2, b_2 - v_2(s_1 + s_2), 0)}$$

$$\cdot\, \mathcal{U}_{A_1 A_2,\, v_1 + A_1 v_2,\, b_1 - v_1 s_1 + A_1(b_2 - v_2(s_1 + s_2)),\, 0}\; U(-s_1, -s_2)$$

Now RHS

$$= e^{i\omega(A_1,v_1,b_1,s_1,A_2,v_2,b_2,s_2)} \mathcal{U}_{A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2,\, s_1+s_2}$$

$$= e^{i\omega(\cdot,\cdot)} U(-s_1-s_2) \mathcal{U}_{A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2,\, 0}$$

$$= e^{i\omega(\cdot,\cdot)} e^{i\omega_2(A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2,\, -s_1-s_2)}$$

$$\cdot \mathcal{U}_{A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2+(v_1+A_1v_2)(-s_1-s_2),\, 0}\, U(-s_1-s_2)$$

$$= e^{i\omega(\cdot,\cdot)} e^{i\omega_2(A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2,\, -s_1-s_2)}$$

$$\cdot \mathcal{U}_{A_1A_2,\, v_1,\, A_1v_2,\, b_1+A_1b_2-v_1s_1-A_1v_2s_1-A_1v_2s_2,\, 0}\, U(-s_1-s_2)$$

Next, equating the exponents

$$\Rightarrow w_2(A_1,v_1,b_1,-s_1) + w_2(A_2,v_2,b_2,-s_1-s_2) + w_2(A_1,v_1,b_1-v_1s_1,0,A_2,v_2,b_2$$

$$- v_2(s_1+s_2),0) = w(A_1,v_1,b_1,s_1,A_2,v_2,b_2,s_2)$$

$$+ w_2(A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2,\, -s_1-s_2)$$

$$\Rightarrow w(A_1,v_1,b_1,s_1,A_2,v_2,b_2,s_2) = w_2(A_1,v_1,b_1,-s_1) + w_2(A_2,v_2,b_2,-s_1-s_2)$$

$$+ w_2(A_1,v_1,b_1-v_1s_1,0,A_2,v_2,b_2-v_2(s_1+s_2),0)$$

$$- w_2(A_1A_2,\, v_1+A_1v_2,\, b_1+v_1s_2+A_1b_2,\, -s_1-s_2)$$

$$\Rightarrow \omega(A_1, v_1, b_1, s_1, A_2, v_2, b_2, s_2) = \frac{1}{2} m s_1 |v_1|^2 + \frac{1}{2} m(s_1+s_2)|v_2|^2$$

$$- m(b_1 - v_1 s_1)\cdot(A_1 v_2)$$

$$- \frac{1}{2} m(s_1+s_2)\,|v_1 + A_1 v_2|^2$$

$$= \cancel{\frac{1}{2} m s_1 |v_1|^2} + \cancel{\frac{1}{2} m s_1 |v_2|^2} + \cancel{\frac{1}{2} m s_2 |v_2|^2}$$

$$- m(b_1 A_1 v_2) + \cancel{m s_1 v_1 A_1 v_2}$$

$$- \cancel{\frac{1}{2} m s_1 |v_1|^2} - \frac{1}{2} m s_2 |v_1|^2$$

$$- \cancel{\frac{1}{2} m s_1 |v_2|^2} - \cancel{\frac{1}{2} m s_2 |v_2|^2}$$

$$- \cancel{\frac{1}{2} m(s_1)(2 v_1 A_1 v_2)}$$

$$- \frac{1}{2} m(s_2)(2 v_1 A_1 v_2)$$

$$= \frac{1}{2} m s_2 |v_1|^2 - m(b_1 - v_1 s_2)(A_1 \cdot v_2)$$

explicit form of $\omega(A_1, v_1, b_1, s_1, A_1, v_2, b_2, s_2)$

Problem 3

Corollary 4.13 (Evolution of a Gaussian state)

i) Let $\psi(x) := e^{-|x|^2}$ for $x \in \mathbb{R}^3$. Let $t > 0$

Then $\psi_t(x)$ is also a Gaussian function centered at $x = 0$.

ii) Let $\tilde{\psi} := \mathcal{B}_v \psi$, $v \in \mathbb{R}^3$

Then $\tilde{\psi}_t$ is a Gaussian function centered at $x = vt$

Proof (i) To prove $\psi_t(x)$ is a Gaussian centered at $x = 0$ $\forall t > 0$

$$\psi_t(x) = \frac{e^{-i\frac{3}{4}\pi}}{(2\pi t)^{3/2}} \int e^{i(x-y)^2 \frac{m}{2t}} e^{-y^2} dy$$

$$= \frac{e^{-i\frac{3}{4}\pi}}{(2\pi t)^{3/2}} \int e^{ix^2 \frac{m}{2t}} \cdot e^{-2ixy \frac{m}{2t}} e^{iy^2 \frac{m}{2t}} e^{-y^2} dy$$

$$= \frac{e^{-i(x^2 \frac{m}{2t} - \frac{3}{4}\pi)}}{(2\pi t)^{3/2}} \int e^{-2ixy \frac{m}{2t}} e^{-y^2(1 - \frac{im}{2t})} dy$$

$$= \frac{e^{-i(x^2 \frac{m}{2t} - 3/4\pi)}}{(2\pi t)^{3/2}} \int e^{-2ix(1-\frac{im}{2t})^{-1/2} y (1 - \frac{im}{2t})^{1/2} \frac{m}{2t}} \times e^{-y^2(1-\frac{im}{2t})} dy$$

Let $X = ix\left(1-\frac{im}{2t}\right)^{-1/2}\frac{m}{2t}$

$Y = y\left(1-\frac{im}{2t}\right)^{1/2} \Rightarrow Y^2 = y^2\left(1-\frac{im}{2t}\right)$

$dY = dy\left(1-\frac{im}{2t}\right)^{1/2}$

$$= \frac{e^{-i\left(x^2\frac{m}{2t}-\frac{3}{4}\pi\right)}}{(2\pi t)^{3/2}}\int e^{-2XY}\cdot e^{-Y^2}\,dY$$

$$= \frac{e^{-i\left(x^2\frac{m}{2t}-\frac{3}{4}\pi\right)}}{(2\pi t)^{3/2}}\int e^{X^2}e^{-X^2}e^{-2XY}e^{-Y^2}\,dY$$

$$= \frac{e^{-i\left(x^2\frac{m}{2t}-\frac{3}{4}\pi\right)}}{(2\pi t)^{3/2}}\int e^{X^2}e^{-(X+Y)^2}\,dY$$

$$= \frac{e^{-i\left(x^2\frac{m}{2t}-\frac{3}{4}\pi\right)}}{(2\pi t)^{3/2}}\cdot e^{-x^2\left(1-\frac{im}{2t}\right)\frac{m^2}{4t^2}}$$

$$\times\int e^{-\left(ix\left(1-\frac{im}{2t}\right)^{1/2}\frac{m}{2t}+y\left(1-\frac{m}{2t}\right)^{1/2}\right)^2}dy$$

$$= \frac{e^{\cancel{-\frac{ix^2m}{2t}}-i\frac{3}{4}\pi-\frac{x^2m^2}{4t^2}+\cancel{\frac{ix^2m}{2t}}}}{(2\pi t)^{3/2}}$$

$$\times\int e^{-\left(ix\left(1-\frac{im}{2t}\right)^{1/2}\frac{m}{2t}+y\left(1-\frac{m}{2t}\right)^{1/2}\right)^2}dy$$

$$= \frac{e^{-1/4\left(\frac{x^2 m^2}{t^2} + 3i\pi\right)}}{(2\pi t)^{3/2}} \int e^{-\left(\frac{ixm}{2t}\left(1-\frac{im}{2t}\right)^{-1/2} + y\left(1-\frac{m}{2t}\right)^{1/2}\right)^2} dy$$

Let $Z = \frac{ixm}{2t}\left(1-\frac{im}{2t}\right)^{1/2} + y\left(1-\frac{m}{2t}\right)^2$

$$dZ = \left(1-\frac{m}{2t}\right)^2 dy$$

$$= \frac{e^{-1/4\left(\frac{x^2 m^2}{t^2} + 3i\pi\right)}}{(2\pi t)^{3/2}} \int e^{-|Z|^2} dy$$

$$= \frac{e^{-1/4\left(\frac{x^2 m^2}{t^2} + 3i\pi\right)}}{(2\pi t)^{3/2}(2\pi)^{3/2}} = \frac{1}{(4\pi^2 t)^{3/2}} e^{-\frac{x^2}{\frac{4t^2}{m^2}}} \cdot e^{-3i\pi}$$

still centered at 0

just a complex phase

b) To prove $\tilde{\psi} = B_v \psi$ is a Gaussian centered at $x = vt$

$$\tilde{\psi}_t = U(t)\,\mathcal{B}_v\,\psi$$
$$= \mathcal{U}_{1,0,0,-t}\,\mathcal{U}_{1,v,0,0}\,\psi$$
$$= e^{i\hat{\omega}}\,\mathcal{U}_{1,v,tv,-t}\,\psi$$
$$= \underbrace{e^{i\hat{\omega}}\,\mathcal{B}_v}_{\text{complex plane}}\;\underbrace{T_{tv}\;\underbrace{U(t)\psi}_{\text{(a) Gaussian centered at } 0}}_{\text{Gaussian centered at } x = vt}$$

Hence centered at $x = vt$.