Example for inequality

$$e^{A+B} \neq e^{A} e^{B}$$

Consider $A, B \in \mathbb{R}^2$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for any $K \geq 2$, $A^K = 0$, $B^K = 0$

$$e^A = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$e^B = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$e^A e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2e^0 & 1e^0 \\ 1e^0 & 1e^0 \end{pmatrix}$$

Next,

$$C = A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We diagonalise this matrix $\qquad (A - \lambda I)v = 0$

eigenvalues of $A+B = \{-1, +1\}$

eigenvector for $\lambda = -1$, $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

eigenvector for $\lambda = +1$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C = PDP^{-1}$ is the diagonalization of C

Next $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \frac{1}{2}\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_{P}\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_{P^{-1}}$$

$$e^{A+B} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{-1} & e \\ e^{-1} & e \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-1}+e & -e^{-1}+e \\ -e^{-1}+e & e^{-1}+e \end{pmatrix} = \begin{pmatrix} e^{-1}+e & -e^{-1}+e \\ -e^{-1}+e & e^{-1}+e \end{pmatrix}$$

Hence $e^{A+B} \neq e^{A} \cdot e^{B}$