Soluzione Esercizio VIII

Problema 1: $\displaystyle\int_0^1 \frac{\sqrt{1-x}-\sqrt{x}}{2\sqrt{1-x}\sqrt{x}}\,dx$

Improprio in $x=1$ e in $x=0$.
↪ tagliare in 2 pezzi, a un punto x_0, per esempio $x_0 = 1/2$.

$$= \lim_{h\to 0+}\int_h^{1/2}\frac{\sqrt{1-x}-\sqrt{x}}{2\sqrt{1-x}\sqrt{x}}\,dx + \lim_{\delta\to 0+}\int_{1/2}^{1-\delta}\frac{\sqrt{1-x}-\sqrt{x}}{2\sqrt{1-x}\sqrt{x}}\,dx$$

oppure $\displaystyle\lim_{\delta\to 1-}\int_{1/2}^{\delta}(\dots)\,dx$

$$= \lim_{h\to 0+}\int_h^{1/2}\frac{\cancel{\sqrt{1-x}}}{2\cancel{\sqrt{1-x}}\sqrt{x}}\,dx - \lim_{h\to 0+}\int_h^{1/2}\frac{\cancel{\sqrt{x}}}{2\sqrt{1-x}\cancel{\sqrt{x}}}\,dx$$

$$+ \lim_{\delta\to 0+}\int_{1/2}^{1-\delta}\frac{\cancel{\sqrt{1-x}}}{2\cancel{\sqrt{1-x}}\sqrt{x}}\,dx - \lim_{\delta\to 0+}\int_{1/2}^{1-\delta}\frac{\cancel{\sqrt{x}}}{2\sqrt{1-x}\cancel{\sqrt{x}}}\,dx$$

$$= \frac{1}{2}\left(\lim_{h\to 0+}\int_h^{1/2}\frac{1}{\sqrt{x}}\,dx - \lim_{h\to 0+}\int_h^{1/2}\frac{1}{\sqrt{1-x}}\,dx + \lim_{\delta\to 0+}\int_{1/2}^{1-\delta}\frac{1}{\sqrt{x}}\,dx - \lim_{\delta\to 0+}\int_{1/2}^{1-\delta}\frac{1}{\sqrt{1-x}}\,dx\right)$$

$$\begin{cases}\int x^{-1/2}\,dx = 2x^{1/2}\\ \int (1-x)^{-1/2}\,dx = -2(1-x)^{1/2}.\end{cases}$$

$= 0.$ (In particolare, è convergente.)

$\displaystyle\int_{1/2}^1 \frac{e^{-\cos(x)}}{\sqrt{1-x}}\,dx$: $-1 \le \cos(x) \le 1 \quad \forall x\in\mathbb{R}$.

$\Rightarrow e^{-1} \le e^{-\cos(x)} \le e \quad \forall x\in\mathbb{R}$.

Criterio del confronto:

$$\int_{1/2}^1 \frac{e^{-\cos(x)}}{\sqrt{1-x}}\,dx \le \int_{1/2}^1\frac{e}{\sqrt{1-x}} = e\int_{1/2}^1\frac{1}{\sqrt{1-x}}\,dx$$

$$= e\int_{1/2}^0 \frac{1}{\sqrt{y}}(-1)\,dy = e\int_0^{1/2}\frac{1}{\sqrt{y}}\,dy < +\infty.$$

Sostituzione:
$y := 1-x$
$dy = -dx$
$x_0 = \frac12 \Rightarrow y_0 = \frac12$
$x_1 = 1 \Rightarrow y_1 = 0.$

$\displaystyle\int_0^1 \frac{1}{y^p}\,dy < +\infty$ se e solo se $p<1$

$y = 1-x$

$dy = \frac{d}{dx}(1-x)\,dx = -dx.$

Problema 2: (1) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^p}$. Usiamo il criterio dell'integrale:

Serie è convergente se e solo se $\displaystyle\int_2^{\infty}\frac{1}{x(\ln(x))^p}\,dx$ è convergente.

$$\int_2^M \frac{1}{x(\ln(x))^p}\,dx = \lim_{M\to+\infty}\int_2^M \frac{1}{x(\ln(x))^p}\,dx$$

Sostituzione: $y := \ln(x)$ $\quad dy = \frac{d}{dx}(\ln(x))\,dx = \frac{1}{x}dx$

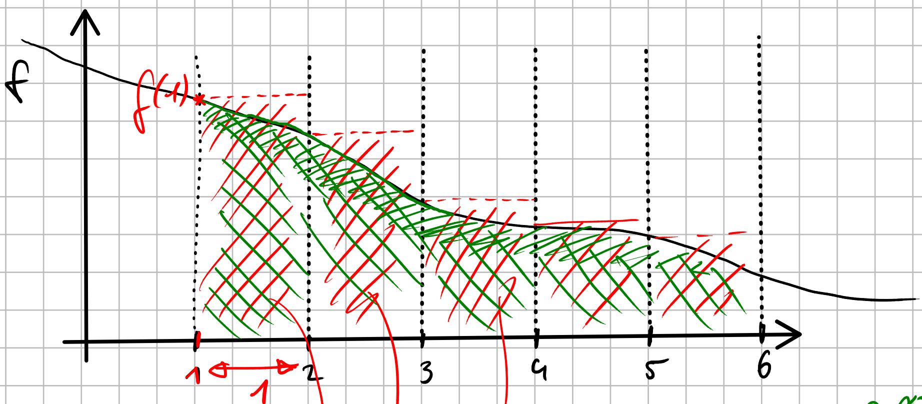
$x_0 = 2 \Rightarrow y_0 = \ln(x_0) = \ln(2)$

$x_1 = M \Rightarrow y_1 = \ln(M)$

$$= \lim_{M\to\infty} \int_{\ln(2)}^{\ln(M)} \frac{1}{y^p}\,dy = \int_{\ln(2)}^{+\infty} \frac{1}{y^p}\,dy$$

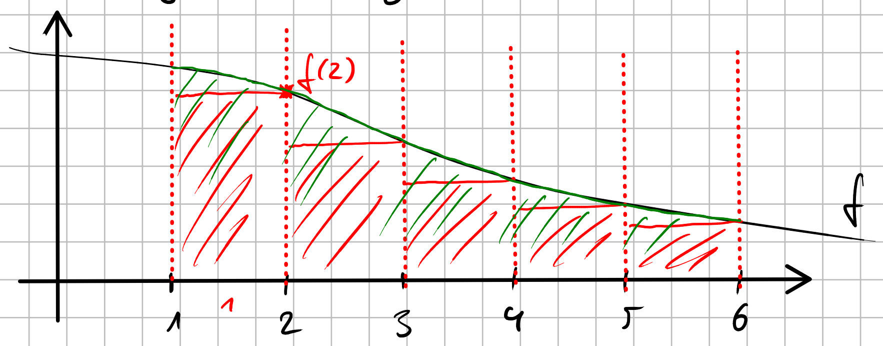
converge se e solo se $p > 1$.

(2) Dimostrazione del Criterio dell'integrale:



$$\sum_{n=1}^{\infty} \underbrace{f(n)\cdot 1}_{\text{area del rettangolo}} \qquad n=1 \quad n=2 \quad n=3 \qquad \geq \int_1^{\infty} f(x)\,dx$$

Se l'integrale diverge, anche la serie diverge.



$$\sum_{n=1}^{\infty} f(n+1)\cdot 1 \leq \int_1^{\infty} f(x)\,dx < +\infty$$

$$= \sum_{n=2}^{\infty} f(n)$$

Se l'integrale converge, anche la serie converge.

Problema 3: formula di Taylor

(1) $n=4$, $x_0=0$, $f(x)=\cos(x)$.

$$\cos(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + R_4(x)$$

Derivate:

$$\left.\begin{aligned} f(0) &= \cos(0) = 1 \\ f'(0) &= -\operatorname{sen}(0) = 0 \\ f''(0) &= -\cos(0) = -1 \\ f'''(0) &= \operatorname{sen}(0) = 0 \\ f^{(4)}(0) &= \cos(0) = 1 \end{aligned}\right\}\ \text{per il polinomio di Taylor}$$

per il resto: $f^{(5)}(x) = -\operatorname{sen}(x)$ non solo nel punto $x=0$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \int_0^x \frac{(x-t)^4}{4!}\,\underbrace{\operatorname{sen}(t)}_{}\,dt$$

$f^{(5)}(t)$ nel punto t!

(2) Dimostrare Taylor:

Taylor: $f(x) = f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^x (x-t)\,f''(t)\,dt$.

$$\underset{\text{ipotesi}}{0} = \lim_{x\to x_0} f(x) = \lim_{x\to x_0}\left(f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^x (x-t)\,f''(t)\,dt\right)$$

$$= f(x_0) + f'(x_0)\underbrace{(x_0-x_0)}_{=0} + \underbrace{\int_{x_0}^{x_0}(x_0-t)\,f''(t)\,dt}_{=0}$$

$$= f(x_0)$$

$\int_a^a f(x)dx = 0$

$\Rightarrow f(x_0) = 0$.

$\Rightarrow$ Taylor: $f(x) = f'(x_0)(x-x_0) + \int_{x_0}^x (x-t)\,f''(t)\,dt$.

Allora

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x_0)(x-x_0) + \int_{x_0}^x (x-t)\,f''(t)\,dt}{g'(x_0)(x-x_0) + \int_{x_0}^x (x-t)\,g''(t)\,dt} \quad \begin{matrix} |:(x-x_0) \\ |:(x-x_0) \end{matrix}$$

$$= \lim_{x\to x_0}\frac{f'(x_0) + \frac{1}{x-x_0}\int_x^{x_0}(x-t)\,f''(t)\,dt}{g'(x_0) + \frac{1}{x-x_0}\int_x^{x_0}(x-t)\,g''(t)\,dt}.$$

(4)

Se possiamo dimostrare $\lim_{x\to x_0} \frac{1}{x-x_0}\int_x^{x_0}(x-t)f''(t)\,dt = 0$ (*)

troviamo: $= \frac{f'(x_0)}{g'(x_0)}$ L'Hôpital ✓. $\left(= \lim_{x\to x_0}\frac{f'(x)}{g'(x)}\right.$ perché f', g' sono continue$\left.\right)$

Come dimostrare (*)?

$$\lim_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^{x}(x-t)f''(t)\,dt$$

$$= \lim_{x\to x_0}\left(\frac{1}{x-x_0}\int_{x_0}^{x}(x-t)f''(t)\,dt - \frac{1}{x-x_0}\int_{x_0}^{x}(\underline{x_0}-t)f''(t)\,dt + \frac{1}{x-x_0}\int_{x_0}^{x}(\underline{x_0}-t)f''(t)\,dt\right)$$

inserito uno zero

$$= \lim_{x\to x_0}\left(\cancel{\frac{1}{x-x_0}}\int_{x_0}^{x}\cancel{((x-t)-(x_0-t))}f''(t)\,dt\right) + \lim_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^{x}(x_0-t)f''(t)\,dt$$

$$= \lim_{x\to x_0}\int_{x_0}^{x}f''(t)\,dt + \underbrace{\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h}(x_0-t)f''(t)\,dt}$$

formula fondamentale del calcolo integrale

$$\hookrightarrow\ = (x_0-t)f''(t)\Big|_{t=x_0} = 0.$$

$$= \lim_{x\to x_0}\big[f'(t)\big]_{x_0}^{x}$$

$$= \lim_{x\to x_0}\big(f'(x)-f'(x_0)\big)$$

$$= 0.$$

(3) $\lim_{x\to 0}\frac{\ln(\cos(x))}{x^2}$

Visto a lezione:

$$\cos(x) = 1 - \frac{x^2}{2!} + o(x^3)$$

$$\ln(1+y) = y - \frac{y^2}{2} + o(y^{2.5})$$

Perché $o(y^{2.5})$ e non $o(y^3)$?

Ricordiamo: $\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + \dots$

$f(x) = o(g(x)) \Leftrightarrow \lim_{x\to 0}\frac{f(x)}{g(x)} = 0$

$y^3 = o(y^3)$? $\lim_{y\to 0}\frac{y^3}{y^3} = 1$

$y^3 = o(y^{2.5})$? $\quad \lim_{y\to 0} \frac{y^3}{y^{2.5}} = \lim_{y\to 0} y^{1/2} = \lim_{y\to 0} \sqrt{y} = 0$

$$\ln(\cos(x)) = \ln\Big(1 \underbrace{- \frac{x^2}{2!} + o(x^3)}_{=:\, y}\Big)$$

$$= y - \frac{y^2}{2} + o(y^{2.5})$$

$$= \left(-\frac{x^2}{2!} + o(x^3)\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + o(x^3)\right)^2 + o\Big(\underbrace{(x^2)^{2.5}}_{x^{2\cdot\frac{5}{2}} = x^5}\Big)$$

$$= -\frac{x^2}{2!} + o(x^3)$$

$$\Rightarrow \lim_{x\to 0} \frac{\ln(\cos(x))}{x^2} = \lim_{x\to 0} \frac{-\frac{x^2}{2} + o(x^3)}{x^2}$$

$$= \lim_{x\to 0}\left(-\frac{1}{2} + o(x)\right) = -\frac{1}{2}.$$

$\blacksquare$

$o(x^n)$ **assorbe** tutti i termini x^m con $m > n$
(per $x \to 0$)

Esempio: $\quad o\left(\left(-\frac{1}{2}x^2 + o(x^3)\right)^{5/2}\right) = o\left(\left(o(x^{1.9})\right)^{5/2}\right)$

$$= o\left(x^{1.9\cdot\frac{5}{2}}\right) = o(x^3)$$

(oppure anche x^4)

$x^2 = o(x^2)$ non è vero!
$\lim_{x\to 0} \frac{x^2}{x^2} = 1 \neq 0$

$x^2 = o(x^{1.9})$ è vero:
$\lim_{x\to 0} \frac{x^2}{x^{1.9}} = \lim_{x\to 0} x^{0.1} = 0$

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(4) $\longrightarrow$ libro pagina 252.