# Metodi Matematici della Meccanica Quantistica

## Solutions for Assignment 1

Discussed on **Friday, October 6, 2023**.
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### Problem 1: Banach Spaces and Bounded Operators (2+2+3+3 points)

**a.** We have to verify the three axioms of a norm:

- *Homogeneity*: $\|\lambda[f]\|_p = \left(\int |\lambda f(x)|^p \mathrm{d}x\right)^{1/p} = \left(|\lambda|^p \int |f(x)|^p \mathrm{d}x\right)^{1/p}$
$= |\lambda| \left(\int |f(x)|^p \mathrm{d}x\right)^{1/p} = \lambda \|[f]\|_p$

- *Positive definiteness*: Since $|f(x)| \geq 0$, we also have $\int |f(x)|^p \mathrm{d}x \geq 0$ and thus $\|[f]\|_p \geq 0$. The case $\|[f]\|_p = 0$ only occurs if[1]

$$\int |f(x)|^p \mathrm{d}x = 0 \quad \Leftrightarrow \quad |f(x)|^p = 0 \text{ a.e.} \quad \Leftrightarrow \quad f(x) = 0 \text{ a.e.} \quad \Leftrightarrow \quad [f] = 0$$

- *Triangle inequality*: This proof is a bit more tricky. It uses Hölder's inequality

$$\|[fg]\|_1 \leq \|[f]\|_p \|[g]\|_q, \tag{1}$$

which holds for any $[f] \in L^p(\mathbb{R}^d), [g] \in L^q(\mathbb{R}^d)$, where $1 = \frac{1}{p} + \frac{1}{q} \Leftrightarrow \frac{p}{q} = p - 1$ and $[fg](x) := f(x)g(x)$ (a.e.) is the pointwise product.
What we have to show is $\|[f] + [g]\|_p \leq \|[f]\|_p + \|[g]\|_p$. For $p > 1$,

$$\begin{aligned}
\|[f] + [g]\|_p^p &= \int |f(x) + g(x)|^p \mathrm{d}x = \int |f(x) + g(x)||f(x) + g(x)|^{p-1} \mathrm{d}x \\
&\leq \int |f(x)||f(x) + g(x)|^{p-1} \mathrm{d}x + \int |g(x)||f(x) + g(x)|^{p-1} \mathrm{d}x \\
&\overset{(1)}{\leq} \left(\int |f(x)|^p \mathrm{d}x\right)^{1/p} \left(\int |f(x) + g(x)|^{q(p-1)} \mathrm{d}x\right)^{1/q} \\
&\quad + \left(\int |g(x)|^p \mathrm{d}x\right)^{1/p} \left(\int |f(x) + g(x)|^{q(p-1)} \mathrm{d}x\right)^{1/q} \\
&= \|[f]\|_p \|[f] + [g]\|_p^{p/q} + \|[g]\|_p \|[f] + [g]\|_p^{p/q} \\
&= \left(\|[f]\|_p + \|[g]\|_p\right) \|[f] + [g]\|_p^{p/q} .
\end{aligned}$$

[1] Recall that "a.e." is the abbreviation for "almost everywhere", meaning "for all $x \in \mathbb{R}^d \setminus \mathcal{N}$ where $\mathcal{N}$ is some set of Lebesgue measure zero". Also recall that if a function $f$ is zero almost everywhere, then $[f] = [0] = 0$, i.e., it is in the same equivalence class as the zero function and this equivalence class is the zero element of the vector space $L^p(\mathbb{R}^d)$.

We conclude, using $p/q = p-1$,

$$\|[f]+[g]\|_p^{p-p/q} \leq \|[f]\|_p + \|[g]\|_p \quad \Leftrightarrow \quad \|[f]+[g]\|_p \leq \|[f]\|_p + \|[g]\|_p \,,$$

which is what we wanted to show. In case $p=1$, we simply have

$$\|[f]+[g]\|_1 = \int |f(x)+g(x)| \mathrm{d}x \leq \int |f(x)| \mathrm{d}x + \int |g(x)| \mathrm{d}x = \|[f]\|_1 + \|[g]\|_1 \,.$$

□

**b.** $\tilde{X}$ being a Banach space means that every Cauchy sequence $(x_n)_{n=1}^\infty \subset \tilde{X}$ must converge to a limit $x \in \tilde{X}$.
Now, any Cauchy sequence $(x_n)_{n=1}^\infty \subset \tilde{X}$ is also a Cauchy sequence in $X \supset \tilde{X}$. Since $X$ is a Banach space, this Cauchy sequence indeed has a limit $x \in X$. And since $\tilde{X}$ is closed, this limit $x$ must be an element of $\tilde{X}$, which establishes the proof. □

**c.** Suppose, $(A_n)_{n=1}^\infty \subset \mathcal{L}(X,Y)$ is a Cauchy sequence, i.e., $\forall \varepsilon \; \exists N : \forall n,m \geq N$ : $\|A_n - A_m\|_{\mathcal{L}(X,Y)} < \varepsilon$. Our goal is to construct a limit operator $A \in \mathcal{L}(X,Y)$ such that $A_n \to A$ in $\mathcal{L}(X,Y)$. To do so, let us consider any $x \in X$. For $n,m \geq N$, we have

$$\|A_n x - A_m x\|_Y = \|(A_n - A_m)x\|_Y \leq \|A_n - A_m\|_{\mathcal{L}(X,Y)} \|x\|_X \leq \varepsilon \|x\|_X,$$

which becomes arbitrarily small as $\varepsilon \to 0$. So $(A_n x)_{n=1}^\infty$ is a Cauchy sequence in $Y$. Since $Y$ is a Banach space, there exists a limit $A_n x \to y_x \in Y$. We now define the operator $A : X \to Y$ via $Ax := y_x$ for any $x \in X$ and claim that it is the desired limit of $(A_n)_{n=1}^\infty$.
First, $A$ is bounded, so $A \in \mathcal{L}(X,Y)$, since for any $x \in X$,

$$\|Ax\|_Y = \| \lim_{n\to\infty} A_n x\|_Y \leq \lim_{n\to\infty} \|A_n x\|_Y \leq \limsup_{n\to\infty} \|A_n\|_{\mathcal{L}(X,Y)} \|x\|_X \,.$$

So $\|A\|_{\mathcal{L}(X,Y)} \leq \limsup_{n\to\infty} \|A_n\|_{\mathcal{L}(X,Y)}$ and the latter is bounded as $(A_n)_{n=1}^\infty$ is a Cauchy sequence.
Second, $(A_n)_{n=1}^\infty$ indeed converges to $A$, as for $n > N$,

$$\|(A - A_n)x\|_Y \leq \limsup_{m\to\infty} \|(A_m - A_n)x\|_Y \leq \varepsilon \|x\|_X \,.$$

So $\|A - A_n\|_{\mathcal{L}(X,Y)} \leq \varepsilon$, which can be achieved for any $\varepsilon > 0$. Thus, $A_n \to A$ in $\mathcal{L}(X,Y)$ and the latter space is closed and therefore a Banach space. □

**d.** Our goal is to extend $A$ to any $x \in X \setminus D$. Since $D \subset X$ is dense, there exists a sequence $(x_n)_{n=1}^\infty \subset D$ with $x_n \to x$. As $(x_n)_{n=1}^\infty$ converges, it is in particular a Cauchy sequence. Since (with $\| \,.\, \| = \| \,.\, \|_{\mathcal{L}(X,Y)}$)

$$\|Ax_n - Ax_m\|_Y \leq \|A\| \|x_n - x_m\|_X \,,$$

the sequence $(Ax_n)_{n=1}^\infty \subset Y$ is also a Cauchy sequence. And as $Y$ is a Banach space, there exists a limit $Ax_n \to y_x \in Y$. We now define the extension $\overline{A} : X \to Y$ as

$$\overline{A}x := \begin{cases} Ax & \text{if } x \in D \\ y_x & \text{if } x \in X \setminus D \end{cases} .$$

It remains to prove that $\|\overline{A}\| = \|A\|$. First,

$$\|\overline{A}\| = \sup_{x \in X\setminus\{0\}} \frac{\|\overline{A}x\|}{\|x\|} \geq \sup_{x \in D\setminus\{0\}} \frac{\|Ax\|}{\|x\|} = \|A\| .$$

On the other hand, for $x \in X \setminus D$, we have

$$\|\overline{A}x\|_Y = \|\lim_{n\to\infty} Ax_n\|_Y = \lim_{n\to\infty} \|Ax_n\|_Y \leq \|A\| \lim_{n\to\infty} \|x_n\|_X = \|A\|\|x\|_X .$$

So $\|\overline{A}\| \leq \|A\|$, which finishes the proof. □

**Problem 2: Derivative Operator (5+5 points)**

**a.** To show that $2\pi i\mathbb{Z} \subset \sigma_{\mathrm{p}}(A_3)$, we construct an explicit eigenfunction for every eigenvalue $\lambda_p := 2\pi i p, p \in \mathbb{Z}$. In fact, for $f_p(x) := e^{2\pi i p x}$, we have $f_p(0) = 1 = f_p(1)$ so $f_p \in D_3$, and

$$(A_3 f_p)(x) = f_p'(x) = 2\pi i p e^{2\pi i p x} = \lambda_p f_p(x) . \tag{2}$$

So $f_p$ is indeed an eigenfunction for $\lambda_p$.
We may now finish the proof by showing that $z \in \rho(A_3)$ for any $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$, since then $2\pi i\mathbb{Z} \supset \sigma(A_3) \supset \sigma_{\mathrm{p}}(A_3)$. To do so, we construct the resolvent $(A_3 - z)^{-1}$ explicitly: It is defined on $g \in C([0,1])$ whenever there exists an $f \in D_3$, $f =: (A_3 - z)^{-1} g$ with

$$(A_3 - z) f = g \quad \Leftrightarrow \quad f'(x) - z f(x) = g(x) \ \forall x \in [0,1] . \tag{3}$$

This is an ODE, whose most general solution, e.g., obtained by the method of Green's functions (also called "variation of the constant" or "Duhamel's formula"), reads

$$f(x) = \int_0^x e^{z(x-t)} g(t) \, \mathrm{d}t + f_0 e^{zx} =: (Sg)(x) , \tag{4}$$

with an arbitrary $f_0 \in \mathbb{C}$. Indeed, one can check that

$$f'(x) = \int_0^x z e^{z(x-t)} g(t) \, \mathrm{d}t + (e^{z(x-x)} g(x)) + z f_0 e^{zx} = z f(x) + g(x) , \tag{5}$$

so $f'(x) = z f(x) + g(x)$ is continuous, whence $f \in C([0,1])$. Further, we can attain

$$f(0) = f(1) \quad \Leftrightarrow \quad f_0 = \int_0^1 e^{z(1-t)} g(t) \, \mathrm{d}t + f_0 e^z$$

by choosing $f_0 := (1-e^z)^{-1} \int_0^1 e^{z(1-t)} g(t)\,\mathrm{d}t$. Note that $(1-e^z)^{-1}$ only exists because $z \notin 2\pi i \mathbb{Z}$ (otherwise, the resolvent would be ill-defined). With this choice of $f_0$ we indeed have $f \in D_3$ and $f$ satisfies (3), which is $(A_3 - z)Sg = g$. The operator $S$ is also bounded, as

$$\begin{aligned} |(Sg)(x)| \leq& \int_0^1 |e^z| \max_{t\in[0,1]} |g(t)|\,\mathrm{d}t \leq |e^z| \|g\|_{C([0,1])} ,\\ |(Sg)'(x)| =& |z(Sg)(x) + g(x)| \leq (|z||e^z| + 1)\|g\|_{C([0,1])} . \end{aligned}$$

Further, integration by parts yields

$$S(A_3 - z)f = \int_0^x e^{z(x-t)}(f'(t) - zf(t))\,\mathrm{d}t = [e^{z(x-t)}f(t)]_{t=0}^x = f(x) .$$

So $S$ is indeed the desired resolvent $(A_3 - z)^{-1}$. □

**b.** First we show $\sigma_{\mathrm{p}}(A_4) = \emptyset$, that is, there are no eigenfunctions. Suppose that $f \in D_4$ was an eigenfunction of some eigenvalue $\lambda \in \mathbb{C}$. Then, $f$ solves the Cauchy problem

$$\begin{cases} f'(x) &= \lambda f(x) \quad \text{for } x \in [0,1] \\ f(0) &= 0 \end{cases} ,$$

which, by the Picard-Lindelöf theorem, has the unique solution $f(0) = 0$. So $f$ is the zero function, which can never be an eigenfunction.

To prove $\sigma(A_4) = \mathbb{C}$, we show that for any $z \in \mathbb{C}$, there is no bounded resolvent $(A_4 - z)^{-1}$. In analogy to (3), such a resolvent would only exist if for any $g \in C([0,1])$, there is some $f \in D_4$ with $f' - zf = g$. Recall (4) that the most general solution to this ODE reads

$$f(x) = \int_0^x e^{z(x-t)} g(t)\,\mathrm{d}t + f_0 e^{zx} .$$

Now $f \in D_4$ entails the two conditions $f(0) = 0 \Rightarrow f_0 = 0$ and

$$f(1) = \int_0^1 e^{z(1-t)} g(t)\,\mathrm{d}t = 0 .$$

It is easy to see that the latter condition is violated for some $g \in C([0,1])$, for instance, say

$$g(t) := e^{-z(1-t)} \quad \Rightarrow \quad f(1) = \int_0^1 1\,\mathrm{d}t = 1 \neq 0 .$$

So a resolvent can for no $z \in \mathbb{C}$ be defined on every $g \in C([0,1])$. □

**Problem 3: Operator-valued analytic functions (10 points)**

The assumption that $L : \mathbb{C} \to \mathcal{L}(X)$ is an operator-valued analytic function means that for any $y \in X^*, x \in X$, the function $f_{y,x} : \mathbb{C} \to \mathbb{C}, f_{y,x}(z) := \langle y, L(z)x \rangle$ is analytic. By $\|L(z)\| \leq M$ (which holds uniformly in $z \in \mathbb{C}$) and the Cauchy-Schwarz inequality, we conclude

$$|f_{y,x}(z)| = |\langle y, L(z)x \rangle| \leq \|y\|_{X^*} \|L(z)x\|_X \leq M \|y\|_{X^*} \|x\|_X \ , \tag{6}$$

so $f_{y,x}$ is bounded. Thus, Liouville's theorem applies and $f_{y,x}$ is constant for any fixed $y \in X^*, x \in X$.

From this we now conclude that $L(z)$ is constant, that is, $L(z)x = L(z')x$ for any $z, z' \in \mathbb{C}$ and $x \in X$: We know that for any $y \in X^*$,

$$f_{y,x}(z) = f_{y,x}(z') \quad \Leftrightarrow \quad \langle y, (L(z)x - L(z')x) \rangle = 0 \ ,$$

so $L(z)x - L(z')x = 0$. □