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Metodi Matematici della Meccanica Quantistica

Solutions for Assignment 2

Discussed on **Friday, October 6, 2023**.
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Problem 1: Operator Adjoints (5+5 points)

a. To show: $A \subset B \Rightarrow B^* \subset A^*$**.**

Let $y \in D(B^*)$. By definition of B^* there exists $y' = B^*y$ such that

$$\langle y, Bx \rangle = \langle y', x \rangle \quad \text{for all } x \in D(B) \; .$$

Since $D(A) \subset D(B)$, this is in particular true for all $x \in D(A)$. For $x \in D(A)$ we also have $Bx = Ax$. So

$$\langle y, Ax \rangle = \langle y', x \rangle \quad \text{for all } x \in D(A) \; .$$

Thus $y \in D(A^*)$ and $y' = A^*y$, as was to be shown. □

b. To show: $A \subset B \subset A^*$**.**

Since $B = B^*$, by the previous exercise $A \subset B = B* \subset A^*$. □

Problem 2: On Proposition 2.7 (4+2 points)

a. Just expand the squares from the norms:

$$\begin{aligned}
& \|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2 \\
= \;& \langle x+y, x+y \rangle - \langle x-y, x-y \rangle - i\langle x+iy, x+iy \rangle + i\langle x-iy, x-iy \rangle \\
= \;& \|x\|^2 + \langle x,y \rangle + \langle y,x \rangle + \|y\|^2 \\
& -\|x\|^2 + \langle x,y \rangle + \langle y,x \rangle - \|y\|^2 \\
& -i\|x\|^2 + \langle x,y \rangle - \langle y,x \rangle - i\|y\|^2 \\
& +i\|x\|^2 + \langle x,y \rangle - \langle y,x \rangle + i\|y\|^2 \\
= \;& 4\langle x,y \rangle \; .
\end{aligned}$$

□

b. To show: $\langle Hu, u \rangle = \langle u, Hu \rangle \; \forall u \in D(H)$ **implies that** H **is symmetric.**

We define the quadratic forms

$$\langle \cdot,\cdot\rangle_1 : D(H)\times D(H) \to \mathbb{C}, \qquad \langle u,v\rangle_1 := \langle Hu, v\rangle \,,$$
$$\langle \cdot,\cdot\rangle_2 : D(H)\times D(H) \to \mathbb{C}, \qquad \langle u,v\rangle_2 := \langle u, Hv\rangle \,.$$

Since $\langle Hu,u\rangle = \langle u,Hu\rangle$, both induce the same seminorm[1]:

$$\|u\|_1^2 = \langle u,u\rangle_1 = \langle Hu,u\rangle = \langle u,Hu\rangle = \langle u,u\rangle_2 = \|u\|_2^2 \,.$$

The polarization identity holds by the same computation as before, implying

$$\begin{aligned}
\langle Hu,v\rangle &= \langle u,v\rangle_1 \\
&= \frac{1}{4}\left(\|u+v\|_1^2 - \|u-v\|_1^2 - i\|u+iv\|_1^2 + i\|u-iv\|_1^2\right) \\
&= \frac{1}{4}\left(\|u+v\|_2^2 - \|u-v\|_2^2 - i\|u+iv\|_2^2 + i\|u-iv\|_2^2\right) \\
&= \langle u,v\rangle_2 = \langle u,Hv\rangle.
\end{aligned}$$

□

Problem 3: Still on Proposition 2.7 (4 points)

To show: if $H \subset H^*$, then $\|\psi(t)-\varphi(t)\| = \|\psi(0)-\varphi(0)\| \;\forall t\in\mathbb{R}$.

Let $u(t) := \psi(t)-\varphi(t)$. By the product rule

$$\begin{aligned}
&\frac{\partial}{\partial t}\|u(t)\|^2 = \left\langle \frac{\partial}{\partial t}u(t), u(t)\right\rangle + \left\langle u(t), \frac{\partial}{\partial t}u(t)\right\rangle \\
&= \langle -iHu(t), u(t)\rangle + \langle u(t), -iHu(t)\rangle = i\langle u(t), Hu(t)\rangle - i\langle u(t), Hu(t)\rangle = 0 \,.
\end{aligned}$$

So $\|u(t)\|^2$ is constant. □

Problem 4: Operator with trivial adjoint (3+4+3 points)

a. Since f has compact support, the sum is finite. □

b. To show: $f \mapsto \langle g, Af\rangle$ is not continuous for $g \neq 0$.

Compute

$$\langle g, Af_k\rangle = \langle f_k, \sum_{n=0}^{\infty} f_k(n) e_n\rangle = \sum_{n=0}^{\infty} f_k(n)\langle g, e_n\rangle \,.$$

Since $g \neq 0$ and $(e_n)_n$ an ONB, there exists at least one $n_0 \in \mathbb{N}$ such that $\langle g, e_{n_0}\rangle \neq 0$.

[1] Note that $\|\cdot\|_1$ is a norm if and only if $\mathrm{Ker}(H) = \{0\}$. Otherwise, we may have $\|u\|_1 = 0$ for $u \neq 0$, so it is only a seminorm. The same holds for $\|\cdot\|_2$.

We construct a sequence $(f_k)_{k\in\mathbb{N}}$ such that

$$\|f_k\|_{L^2} \to 0$$

while

$$f_k(n) = f_0(n) \quad \text{for all } n \in \mathbb{N} \; .$$

The latter implies that $\langle g, Af_k\rangle$ is constant with respect to k and non-zero. (It if was continuously depending on f_k, it would have to converge to zero instead.)

We take f_k as a smooth bump function with maximum $f_k(n_0) = 1$ and supported in the interval $[n_0 - \frac{1}{k}, n_0 + \frac{1}{k}]$. □

c. To show: $D(A^*) = \{0\}$.

Assume $D(A^*)$ contains any $g \neq 0$. Then there exists $g' = A^*g$ with

$$\langle g', f\rangle = \langle g, Af\rangle \qquad \forall f \in D(A) \; .$$

Then $f \mapsto \langle g', f\rangle$ is continuous. Contradiction to previous exercise. □