Metodi Matematici della Meccanica Quantistica

Assignment 3

To be handed in on **Wednesday, November 8, 2023, before 10:30** via email (scanned or LaTeX) to sascha.lill@unimi.it or on paper at the beginning of the lecture.

Problem 1: Proof of the Weyl criterion (10 points)

Let X be a Banach space, $A : D \subset X \to X$ an operator and $\lambda \in \mathbb{C}$.

Prove that: If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in D with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, and $\|(A - \lambda)x_n\| \to 0$ $(n \to \infty)$, then $\lambda \in \sigma(A)$.

Problem 2: Coulomb potential (5+5 points)

a. Suppose that $V \in L^2 + L^\infty(\mathbb{R}^3)$.
Show that the L^2-part can be made arbitrarily small, i. e., for every $\varepsilon > 0$ there exist $V_2^\varepsilon \in L^2(\mathbb{R}^3)$ and $V_\infty^\varepsilon \in L^\infty(\mathbb{R}^3)$ such that $V = V_2^\varepsilon + V_\infty^\varepsilon$ and $\|V_2^\varepsilon\|_2 < \varepsilon$.

b. Show that the Coulomb potential, defined by $V(x) := \frac{1}{|x|}$ for $x \neq 0$ (and $V(0) := 0$), is in $L^2 + L^\infty(\mathbb{R}^3)$.

Problem 3: Sobolev inequalities (5+5 points)

a. Assume that for functions defined on $\mathbb{R}^n$ a Sobolev inequality of the following form holds: There exists a constant $C_{n,p,q}$ such that for all f we have

$$\|f\|_{L^q} \leq C_{n,p,q} \|\nabla f\|_{L^p} \ .$$

Given n and p, consider a rescaling $f_\lambda(x) = f(\lambda x)$ by a parameter $\lambda > 0$ to determine the only possible exponent q for which this can hold.

b. Let $u \in H^m(\mathbb{R}^n)$ with $m > n/2$ (the Sobolev space such that u has m weak derivatives in L^2). Use the Fourier transform to show that $u \in L^\infty(\mathbb{R}^n)$ and

$$\|u\|_{L^\infty} \leq C_{m,n} \|u\|_{H^m} \ , \tag{1}$$

where the constant $C_{m,n}$ does not depend on u.

Problem 4: On the Existence Proof for SCUGs (5+5 points)

Let $A : D \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

In the lecture we defined $B_m := im(A + im)^{-1}$ ($m \in \mathbb{Z}$), $A_m := B_m A B_{-m}$, and

$$U_m(t) := e^{-iA_m t} := \sum_{k \in \mathbb{N}} \frac{1}{k!} (-itA_m)^k \ .$$

a. Show that $U_m(t)$ is a SCUG for every $m \in \mathbb{N}$. Then show that the limit of $U_m(t)\varphi$ ($m \to \infty$) exists for all $\varphi \in D$ and all $t \in \mathbb{R}$. Why does $U(t) := \text{s-lim}_{m\to\infty} U_m(t)$ exist?

b. Show that $U(t)$ is a SCUG. Then show that its generator is A.

Problem 5: Resolvent of the Laplacian (10 points)

Let $H_0 = -\Delta$ (in $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$). Show that for $\varphi \in L^2(\mathbb{R}^3)$ and $\kappa > 0$

$$\left(\left(H_0 + \kappa^2\right)^{-1} \varphi \right)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} \varphi(y) \mathrm{d}y \ .$$

Hints: Recall that a multiplication operator under Fourier transform turns into convolution with the inversely transformed function $\check{f}$ (why?):

$$\left(\mathcal{F}^{-1} T_f \mathcal{F} \varphi\right)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \check{f}(x-y)\varphi(y)\mathrm{d}y \ .$$

Then use spherical coordinates and the residue theorem of complex analysis (you can look it up) to calculate $\check{f}$.

Remark: The function $\frac{e^{-\kappa|x|}}{4\pi|x|}$ is called Yukawa potential or free Green's function. In electrodynamics or partial differential equations, one considers linear differential operators L and tries to find a so called Green's function G such that $(LG)(x) = \delta(x)$ (Dirac delta distribution). So you can think of a Green's function G as the representation of a resolvent by an integral operator.