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Metodi Matematici della Meccanica Quantistica

Assignment 4

To be handed in on **Wednesday, November 22, 2023, before 10:30** via email (scanned or LaTeX) to sascha.lill@unimi.it or on paper at the beginning of the lecture.

Problem 1: Method of Stationary Phase (7 + 8 points)

a. Let $\omega \in C^\infty(\mathbb{R}^n)$ and $u \in C_0^\infty(\mathbb{R}^n)$. Let G be an open neighbourhood of $\{\nabla\omega(k) \mid k \in \operatorname{supp}(u)\}$.

Show — for simplicity only in the case of space dimension $n = 1$ — that for all $m \in \mathbb{N}$ there exists $c_m \in \mathbb{R}$ such that

$$\left| \int_{\mathbb{R}^n} e^{i(k\cdot x - \omega(k)t)} u(k) \mathrm{d}k \right| \leq \frac{c_m}{(1+|t|)^m} \quad \text{for all } x \text{ and } t \text{ satisfying } \frac{x}{t} \notin G.$$

Heuristics: This quantifies the intuition that positive and negative parts of the integral cancel if the phase is rapidly oscillating.

Hint: Let $\phi(k) := k\frac{x}{t} - \omega(k)$. Use

$$e^{i\phi(k)t} = \left(\frac{1}{i\phi'(k)t} \frac{\mathrm{d}}{\mathrm{d}k} \right)^m e^{i\phi(k)t}$$

and integration by parts.

b. Let $\omega(p) := \sqrt{p^2+1}$ and consider the pseudo-relativistic Hamiltonian

$$H = H_0 + V \ , \quad H_0 = \sqrt{-\Delta + 1} := \mathcal{F}^{-1} T_\omega \mathcal{F}$$

where the potential $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies, for some $\mu > 1$, the decay $|V(x)| \leq \text{const} \cdot |x|^{-\mu}$ for $|x| > R$. You can take for granted that $H = H^*$ on $D(H_0)$.

Show that the wave operators $\Omega_\pm$ exist.

Hint: Let $D = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid \hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\}$. Let $\varphi \in D$ with $\operatorname{supp}(\hat{\varphi}) \subset \{|p| \geq \varepsilon\}$. Then for all $p \in \operatorname{supp}(\hat{\varphi})$ we have by monotonicity

$$|\nabla\omega(p)| = \frac{|p|}{\sqrt{p^2+1}} \geq \frac{\varepsilon}{\sqrt{\varepsilon^2+1}} =: 2\delta \ .$$

Decompose the potential into parts $|x| \leq \delta t$ and $|x| > \delta t$.

Problem 2: Abelian Limits (5 + 5 points)

a. Let $H_0 = -\Delta/2$ in $L^2(\mathbb{R}^3)$ and $E \in \mathbb{R}$. Show that

$$\operatorname*{s-lim}_{\varepsilon \downarrow 0} \varepsilon (H_0 - E + i\varepsilon)^{-1} = 0.$$

Hint: Use the Fourier transform and choose a convenient dense subspace.

b. Let $\varphi : [0, \infty) \to X$ be continuous, X a Banach space and assume that $\varphi_\infty := \lim_{t\to\infty} \varphi(t)$ exists. Prove that

$$\varphi_\infty = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} \varphi(t) \mathrm{d}t \ .$$

Problem 3: Compact Operators (2 + 3 points)

a. Let X, Y, Z be Banach spaces. Let $K \in \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(Y, Z)$ (i. e., bounded operators). Let moreover K or L be compact. Show that in both cases LK is compact.

b. Show that if K is a finite-rank operator, then it is also compact.