Prof. Dr. Niels Benedikter
Dr. Sascha Lill

Metodi Matematici della Meccanica Quantistica

Assignment 5

To be handed in on **Wednesday, December 13, 2023, before 10:30** via email (scanned or LaTeX) to sascha.lill@unimi.it or on paper at the beginning of the lecture.

Problem 1: $f(x)g(-i\nabla)$ (10 points)

Let $f, g \in L^2(\mathbb{R}^n)$ and let $B := f(x)g(-i\nabla)$. Show that $\|B\| \leq (2\pi)^{-n/2}\|f\|_{L^2}\|g\|_{L^2}$ and $\|B\| \leq \|f\|_{L^\infty}\|g\|_{L^\infty}$.

Problem 2: Essential Spectrum of $-\Delta$ (10 points)

Consider $-\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Show that $\sigma_{\text{ess}}(-\Delta) \supset [0, \infty)$ by explicitly constructing Weyl sequences in position space.

Problem 3: Weak Compactness (10 points)

Show—without using the Banach–Alaoglu theorem—that any bounded sequence in a separable Hilbert space has a weakly convergent subsequence.

Problem 4: Spectral Projection (10 points)

Let $A = A^*$ a densely defined operator on a Hilbert space $\mathcal{H}$. Let λ be an isolated point of the spectrum $\sigma(A)$. Let $P := \chi_{\{\lambda\}}(A)$ the spectral projection on $\{\lambda\}$ and $P^\perp := 1 - P$. Show that $\lambda \notin \sigma(A \restriction_{P^\perp\mathcal{H}})$.

Problem 5: Approximating Sequence (10 points)

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Show that there exists a sequence of functions $f_N(x)$ with $f_N(x) \to f(x)$ for all $x \in \mathbb{R}^n$ and $\|f_N\|_\infty := \sup_{x\in\mathbb{R}^n}|f_N(x)| \leq \|\hat{f}\|_1 + 1$.