regularity of eigenfunctions
(continued)

EXAMPLE: Let $d = 1$ (space dimension) and consider $H = -\Delta + V$ such that $D(H) = D(-\Delta) = H^2(\mathbb{R})$, e.g., by Kato-Rellich theorem.

Then
$$s > \frac{n}{2} + k, \quad k \in \mathbb{N},$$
$$\begin{matrix} \shortparallel & & \shortparallel \\ 2 & & \frac{1}{2} \end{matrix}$$
permits $k = 1$.

So any eigenfunction φ, $H\varphi = E\varphi$, ($E \in \mathbb{R}$), has to be at least in $C^1(\mathbb{R})$.

In higher dimension we may only get C^0, i.e., continuity.

Where the potential is smooth, we can get stronger statements, for example:

THEOREM 7.4:

Let $H = -\frac{\Delta}{2} + V$ defined on $D(H) = H^2(\mathbb{R}^n)$. Let $\varphi \in D(H)$ with $H\varphi = E\varphi$ for some $E \in \mathbb{R}$.

If $\Omega \subset \mathbb{R}^n$ is open and $V\restriction_\Omega \in C^\infty(\Omega)$, then $\varphi \in C^\infty(\Omega)$.

PROOF:

We will show that $\chi\varphi \in \bigcap_{m \geq 2} H^m(\mathbb{R}^n)$ for all $\chi \in C_0^\infty(\Omega)$.

By the Sobolev lemma $\bigcap_{m \geq 2} H^m(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$, so it follows that $\varphi \in C^\infty(\Omega)$.

We proceed by induction

Base $m=2$:

Since $e \in H^2(\mathbb{R}^n)$, also $\gamma e \in H^2(\mathbb{R}^n)$.

Step m to $m+1$: Let $m \geq 2$, let $\gamma e \in H^m(\mathbb{R}^n)$ for all $\gamma \in C_0^\infty(\Omega)$.

Then

$$(*) \qquad \Delta(\gamma e) = (\Delta\gamma)e + 2\nabla\gamma\cdot\nabla e + \gamma(\Delta e)$$

(use eigenvalue equation)

$$= (\Delta\gamma)e + 2\nabla\gamma\cdot\nabla e + \gamma(V-E)e$$

Writing out the components, we verify that

$$(**) \qquad \nabla\gamma\cdot\nabla e = \operatorname{div}((\nabla\gamma)e) - (\Delta\gamma)e.$$

By assumption $\overset{\text{smooth}}{(\Delta\gamma)}\overset{H^m}{e}$, $\overset{\text{smooth}}{(V-E)}\gamma\overset{H^m}{e}$, and $\underset{\text{smooth}}{(\nabla\gamma)}\underset{H^m}{e}$ are in $H^m(\mathbb{R}^n)$. Next,

$(\nabla\gamma)e \in H^m$ implies $\operatorname{div}((\nabla\gamma)e) \in H^{m-1}(\mathbb{R}^n)$.

Now $(*)$ and $(**)$ together imply

$$\Delta(\gamma e) = \underbrace{(\Delta\gamma)e}_{\in H^m} + 2\underbrace{\operatorname{div}((\nabla\gamma)e)}_{\in H^{m-1}} - 2\underbrace{(\Delta\gamma)e}_{\in H^m} + \underbrace{\gamma(V-E)e}_{\in H^m}$$

So $\Delta(\gamma e) \in H^{m-1}(\mathbb{R}^n)$.

So $\gamma e \in H^{m+1}(\mathbb{R}^n)$.

In Fourier representation this is easy to see:

$\Delta(\chi e) \in H^{m-1}(\mathbb{R}^n)$

$$\Longleftrightarrow \int \left| p^2 \, \widehat{\chi e}(p) \right|^2 (1+p^2)^{m-1} \, dp < +\infty$$

$$\Rightarrow \int |\widehat{\chi e}(p)|^2 (1+p^2)^{m+1} \, dp < +\infty .$$

This concludes the induction. $\blacksquare$

VIII STABILITY OF THE ESSENTIAL SPECTRUM

We start with a quick summary about compact operators.

Def. 8.1: Let X, Y Banach spaces and $K \in \mathcal{L}(X,Y)$. K is called <u>compact</u> if for every bounded sequence in X, $(Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Lemma 8.2: K is compact if and only if, for every bounded set $M \subset X$, $\overline{KM}$ is compact.

Def. 8.3: $K \in \mathcal{L}(X,Y)$ is called <u>finite-rank</u> operator if $\dim \operatorname{ran} K < +\infty$.

Proposition 8.4: Let X, Y, Z Banach spaces.

(a) If K is finite rank, then it is compact.

(b) If K, L are compact and $\alpha, \beta \in \mathbb{C}$, then $\alpha K + \beta L$ is compact.

(c) If $K \in \mathcal{L}(X,Y)$, $L \in \mathcal{L}(Y,Z)$, and K or L is compact, then LK is compact.

(d) If (K_n) is a sequence of compact operators and

operator norm $\|K_n - K\| \longrightarrow 0 \quad (n \to \infty)$,

then K is compact.

(e) If K is compact and $x_n \rightharpoonup x$, then $K x_n \longrightarrow Kx$.

(weak topology)

(norm topology)

Remark:

(a)-(c) mean that compact operators form an ideal.

Easy to prove $\longrightarrow$ Exercises.

We prove the other two statements.

<u>Proof:</u> (d) Let $M \subset X$ be a bounded set.

Set $R := \sup_{x \in M} \|x\|$. Let $\varepsilon > 0$.

Then there exists $n \in \mathbb{N}$ such that

$$\|K_n - K\| < \frac{\varepsilon}{3R}.$$

Since $\overline{K_n M}$ is compact by assumption, there exists a finite covering

$$K_n M \subset \bigcup_{i=1}^{N} B_{\frac{\varepsilon}{3}}(K_n x_i).$$

Let $x \in M$. Now we know that there exists x_i such that

$$\|K_n x - K_n x_i\| < \frac{\varepsilon}{3}.$$

So
$$\begin{aligned} \|Kx - Kx_i\| &\le \|(K-K_n)x\| \\ &\quad + \|K_n x - K_n x_i\| \\ &\quad + \|(K_n - K)x_i\| \\ &< \frac{\varepsilon}{3R} R + \frac{\varepsilon}{3} + \frac{\varepsilon}{3R} \cdot R \le \varepsilon. \end{aligned}$$

Thus $KM \subset \bigcup_{i=1}^{N} B_{\varepsilon}(Kx_i)$.

(e) Let $x_n \longrightarrow x$.
By continuity it follows
$$Kx_n \longrightarrow Kx.$$

Assume $Kx_n \not\longrightarrow Kx$.
Then there exits $\varepsilon > 0$ and a subsequence (x_{n_i}) with
$$\| Kx_{n_i} - Kx \| > \varepsilon \quad \forall i \in \mathbb{N}.$$

But since K is compact,
Kx_{n_i} has a convergent subsequence,
whose limit can only be Kx. (check!

Contradiction ↯. ∎

<u>Proposition 85:</u>
Let $\mathcal{H}$ a separable Hilbert space and $K \in \mathcal{L}(\mathcal{H})$ compact.

Then there exists a sequence of finite-rank operators $(K_n)_{n \in \mathbb{N}}$ with $\| K - K_n \| \longrightarrow 0 \quad (n \to \infty)$.

Proof: Since $\mathcal{H}$ is separable, there exists an ONB $(e_n)_{n\in\mathbb{N}}$.

Define $P_n \in \mathcal{L}(\mathcal{H})$ by

$$P_n\psi := \sum_{i=1}^{n} e_i \langle e_i, \psi\rangle .$$

Then $P_n\psi \longrightarrow \psi \ (n\to\infty)$ and $P_n^* = P_n$.

Set $K_n := K P_n$.

K_n has finite rank.

$$\| K - K_n \| = \| K(1-P_n)\|$$

$$= \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \| K(1-P_n)x\| ,$$

So for any $n\in\mathbb{N}$ there exists $y_n\in\mathcal{H}$ such that $\|y_n\|=1$ and

$$(*) \qquad \|K-K_n\| \leq \|K(1-P_n)y_n\| + \frac{1}{n}.$$

We have, for any $\psi\in\mathcal{H}$:

$$|\langle\psi,(1-P_n)y_n\rangle| = |\langle(1-P_n)\psi, y_n\rangle|$$

$$\le \underbrace{\|(1-P_n)\psi\|}_{\to 0} \; \underbrace{\|y_n\|}_{=1} .$$

Thus $(1-P_n)\,y_n \longrightarrow 0$.

By Proposition 8.4 (e),

$$K(1-P_n)y_n \longrightarrow 0 .$$

By $(*)$: $\|K-K_n\| \longrightarrow 0 \quad (n\to\infty)$.

∎

<u>Def. 8.6:</u> A bounded operator B on $L^2(\mathbb{R}^n)$ is called <u>Hilbert-Schmidt operator</u> if there exists a $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$B\varphi(x) = \int_{\mathbb{R}^n} K(x,y)\,\varphi(y)\,dy .$$

K is called <u>integral kernel</u> of B.

<u>Exercise:</u> By C.S. $\|B\| \le \|K\|_2$.

<u>Def. 8.7:</u> For $\varphi, \psi \in L^2(\mathbb{R}^n)$, define the tensor product by $(\varphi\otimes\psi)(x,y) := \varphi(x)\psi(y)$

<u>Lemma 8.8:</u> If $(e_n)_n$ is an ONB of $L^2(\mathbb{R}^n)$, then $(e_n \otimes e_l)_{n,l}$ is an ONB of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$

<u>Proof:</u> Orthonormality follows from $\langle e_n \otimes e_l, e_r \otimes e_s \rangle = \langle e_n, e_r \rangle \langle e_l, e_s \rangle$ (via Fubini).

Let $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and assume that $\langle K, e_n \otimes e_l \rangle = 0$ for <u>all</u> $n, l \in \mathbb{N}$.

By Fubini this implies

$$\int \underbrace{\left(\int \overline{K(x,y)}\, e_l(y)\, dy \right)}_{=: K_l(x)} e_n(x)\, dx = 0.$$

It follows that $K_l = 0$ in L^2 $\forall l$. That is, for every $l \in \mathbb{N}$ exists a set N_l of measure zero such that $K_l(x) = 0$ on N_l^c.

On $\left(\bigcup_{\ell\in\mathbb{N}} N_\ell\right)^c$ we have:

$$K_\ell(x) = 0 \quad \forall \ell \in \mathbb{N}.$$

$$\|$$

$$\langle K(x,\cdot), e_\ell \rangle_{L^2(\mathbb{R}^n)}.$$

Thus, $\forall x \in \left(\bigcup_{\ell\in\mathbb{N}} N_\ell\right)^c$:

$$\|K(x,\cdot)\|_{L^2} = 0.$$

Countable union of null sets is a null set, so $\bigcup_{\ell\in\mathbb{N}} N_\ell$ is a null set, so

$$\|K\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = \int dx \, \|K(x,\cdot)\|^2_{L^2} = 0.$$

So K cannot be orthogonal to all $e_n \otimes e_\ell$,

i.e. the system of $\{e_n \otimes e_\ell\}$ is maximal as had to be shown. $\blacksquare$

Lemma 8.9: Every Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ is compact.

Proof: Let B a HS operator with kernel K.

Let $(e_n)_{n\in\mathbb{N}}$ a basis of $L^2(\mathbb{R}^n)$.

Set $\mathbb{K}_N := \sum_{i,j=1}^{N} e_i \otimes e_j \langle e_i \otimes e_j, K \rangle$.

Let B_N the HS-operator with kernel $\mathbb{K}_N$.

B_N has finite rank and

$$\|B - B_N\| \leq \|K - K_N\|_{L^2} \longrightarrow 0,$$

since the $(e_k \otimes e_\ell)_{k,\ell}$ form a basis.

By Proposition 8.4 (a)+(d), B is compact. $\blacksquare$

Proposition 8.10: (Kato-Seiler-Simon)

Let $f, g \in L^2(\mathbb{R}^n)$, and

$$B := f(x)\, g(-i\nabla) := T_f\, \mathcal{F}^{-1}\, T_g\, \mathcal{F}$$

as an operator on $L^2(\mathbb{R}^n)$.

Then B is densely defined, bounded, and its extension by continuity is the HS operator with kernel

$$K(x,y) = (2\pi)^{-n/2} f(x)\, \check{g}(x-y).$$

inverse Fourier transform.

We have $\|B\| \leq (2\pi)^{-n/2} \|f\|_{L^2} \|\check{g}\|_{L^2}$

both in operator norm

and $\|B\| \leq \|f\|_{L^\infty} \|g\|_{L^\infty}$.

<u>Proof:</u> Take a domain $\mathcal{S}(\mathbb{R}^n)$.

Then use the convolution theorem: For $\varphi \in \mathcal{S}(\mathbb{R}^n)$:

$$T_f \mathcal{F}^{-1} T_g \mathcal{F} \varphi = T_f \mathcal{F}^{-1}(g \hat{\varphi})$$

$$= T_f (2\pi)^{-n/2} (\mathcal{F}^{-1} g) * (\mathcal{F}^{-1} \hat{\varphi})$$

$$= (2\pi)^{-n/2} T_f (\check{g} * \varphi).$$

To be precise, this holds for $g \in \mathcal{S}(\mathbb{R}^n)$. If $g \in L^2(\mathbb{R}^n)$, approximate it by a sequence $g_n \in \mathcal{S}(\mathbb{R}^n)$ in L^2-norm and then take limit. ▨

Norms: exercise.