Def. 8.11: Let A, B (unbounded) operators with $D(B) \supset D(A)$. If there exists $z \in \varrho(A)$ such that

$$B(A-z)^{-1} \text{ is compact,}$$

we call B *relatively compact* w.r.t. A.

Proposition 8.12: Let $V: \mathbb{R}^n \longrightarrow \mathbb{C}$ measurable, $|V(x)| \longrightarrow 0$ for $|x| \longrightarrow \infty$ and $V \in L^\infty(\mathbb{R}^n)$ or $V \in L^2_{loc}(\mathbb{R}^n)$ with $n \leq 3$.

Then $V(H_0 + 1)^{-1}$ is compact.

Proof: Let

$$V_R(x) := \begin{cases} V(x) & |x| \leq R \\ 0 & |x| > R \end{cases}.$$

Then

$$\| V(H_0+1)^{-1} - V_R(H_0+1)^{-1} \|$$

$$= \| (V - V_R)(H_0+1)^{-1} \|$$

$$\leq \| V - V_R \|_{L^\infty} \| (H_0+1)^{-1} \|$$

$(H_0+1)^{-1}$, as a resolvent, with
$-1 \notin \sigma(H_0) = [0,\infty)$, is bounded
and is independent of R.

As $R \to \infty$ we have $\|V - V_R\|_{L^\infty} \to 0$.

Thus $V_R(H_0+1)^{-1} \longrightarrow V(H_0+1)^{-1}$
in operator norm and it is sufficient
to show that $V_R(H_0+1)^{-1}$ is compact $\forall R > 0$.

L^2_{loc}-case: Let $V \in L^2_{loc}(\mathbb{R}^n)$ with $n \leq 3$.

Of course $V_R \in L^2(\mathbb{R}^n)$.

Moreover with $g(p) := \left(\frac{p^2}{2}+1\right)^{-1}$ we have
$(H_0+1)^{-1} = g(-i\nabla)$.

Thanks to $n \leq 3$, $g \in L^2(\mathbb{R}^n)$.

So by Proposition 8.10, $V_R\, g$ is Hilbert-Schmidt, and thus compact.

L^∞-case: Let $V \in L^\infty(\mathbb{R}^n)$.

Let $g_m(p) := g(p)\,\chi_{\{|p| \leq m\}}$.

Then $g_m \in L^2(\mathbb{R}^n)$,
thus $V_R\, g_m(-i\nabla)$ is Hilbert-Schmidt

Moreover we have convergence in operator norm:

$$\| V_R\, g(-i\nabla) - V_R\, g_m(-i\nabla) \|$$

$$= \| V_R \| \, \| \mathcal{F}^{-1} T_{g-g_m} \mathcal{F} \|$$

$$\leq \| V_R \|_{L^\infty} \, \| T_{g-g_m} \|$$

$$\leq \underbrace{\| V \|_{L^\infty}}_{\substack{<+\infty \\ \text{by assumption}}} \, \underbrace{\| g - g_m \|_{L^\infty}}_{\substack{\to 0 \\ \text{as } m \to \infty.}}$$

Thus $V_R\, g(-i\nabla)$ is compact. ∎

Def. 8.13: Let $A : D(A) \subset \mathfrak{H} \to \mathfrak{H}$, $A = A^*$.

Then the discrete spectrum is

$\sigma_d(A) := \{ \lambda \in \mathbb{C} :$ λ is an eigenvalue of A, λ is isolated in $\sigma(A)$, and the corresponding eigenspace has finite dimension $\}$.

("finite geometric multiplicity of the eigenvalue")

The essential spectrum is

$$\sigma_{ess}(A) := \sigma(A) \setminus \sigma_d(A).$$

Theorem 8.14: (Weyl criterion for the essential spectrum)

Let $A = A^*$. Then $\lambda \in \sigma_{ess}(A)$ if and only if there exists a sequence $(e_n)_n$ in $D(A)$ satisfying the three properties

(i) $\|e_n\| = 1 \quad \forall n \in \mathbb{N}$

(ii) $\|(A-\lambda)e_n\| \longrightarrow 0 \quad (n \to \infty)$

(iii) $e_n \longrightarrow 0 \quad (n \to \infty)$.

Remark: (i) & (ii) characterize $\lambda \in \sigma(A)$, corresponding sequence is called Weyl sequ. (we already saw that much earlier)

A sequ. with (i), (ii), & (iii) is called singular Weyl sequence.

"Example would be if you tried to take plane waves as a Weyl sequ. for $-\Delta$.

$\hookrightarrow$ Exc.

Proof: Let $\lambda \in \sigma_{ess}(A)$.

There are three options:

(a) λ is an eigenvalue of infinite multiplicity

(b) λ has finite multiplicity but is not isolated.

(c) λ is not an eigenvalue.

Case (a): The eig-space has an orthonormal system

$(e_n)_n$, which satisfies $(A-\lambda)e_n = 0$ and like every orthonormal system, $e_n \rightharpoonup 0$.

<u>Case (c):</u> By the known Weyl criterion, for $\lambda \in \sigma(A)$, there exists a sequence $(e_n)_n$ with $\|e_n\| = 1$ and $\|(A-\lambda)e_n\| \longrightarrow 0$.

Any bounded sequence in a separable Hilbert space has a weakly convergent subsequence, so w.l.o.g. $e_n \rightharpoonup e$.

You can either use a diagonal sequence argument as for Arzela-Ascoli, or you rely on the Banach-Alaoglu theorem. (Exc.)

Then for all $\eta \in D(A)$:

$$\begin{aligned}\langle (A-\lambda)\eta, e\rangle &= \lim_{n\to\infty} \langle (A-\lambda)\eta, e_n\rangle \\ &= \lim_{n\to\infty} \langle \eta, (A-\lambda)e_n\rangle \\ &= 0\end{aligned}$$

Thus $(A-\lambda)e = 0$.

But since λ was assumed not to be an eigenvalue, we must have $e = 0$

<u>Case (b):</u> The eigenspace of λ is closed ($A = A^*$ implies that A is closed), so the orthogonal projection P on the eigenspace is well-defined (c.f. Sheet 2, Problem 5b).

Use the eigenvalue equation to verify $PA \subset AP$, and conclude that

$$\sigma(A) = \sigma(A\restriction_{P\mathcal{H}}) \cup \sigma(A\restriction_{P^\perp\mathcal{H}})$$
$$= \{\lambda\} \quad \cup \sigma(A\restriction_{P^\perp\mathcal{H}}).$$

But the spectrum of any operator is closed, so $\sigma(A\restriction_{P^\perp\mathcal{H}})$ is closed, and since λ is not isolated $\Rightarrow \lambda \in \sigma(A\restriction_{P^\perp\mathcal{H}})$.

However, λ is not a eigenvalue of $A\restriction_{P^\perp\mathcal{H}}$.

Now we can apply the argument of case (c).

Direction from Weyl sequence to spectrum:

By contradiction: assume $(e_n)_n$ was a singular Weyl sequence and $\lambda \in \sigma_d(A)$.

Let P the projection on the eigenspace of λ. Since $\lambda \in \sigma_d(A)$, P has finite rank, so $\mathcal{A}$ is compact.

Since $e_n \rightharpoonup 0$ then $Pe_n \to 0$.

Then $\|P^\perp e_n\| \longrightarrow 1$.

For all n large enough: $\|P^\perp e_n\| \geq \frac{1}{2}$.

Let $\varphi_n := \frac{1}{\|P^\perp e_n\|} P^\perp e_n$.

Then $\|\varphi_n\| = 1$ and

$$\|(A-\lambda)\varphi_n\| = \frac{1}{\|P^\perp e_n\|}\|(A-\lambda)P^\perp e_n\|$$

(*) $$\leq 2\,\|(A-\lambda)P^\perp e_n\|$$

$$\begin{aligned}
&\|(A-\lambda)e_n\|^2 \\
&= \|(A-\lambda)(P+P^\perp)e_n\|^2 \\
&= \|(P+P^\perp)(A-\lambda)e_n\|^2 \\
&= \langle (P+P^\perp)(A-\lambda)e_n, (P+P^\perp)(A-\lambda)e_n\rangle \\
&= \langle P(A-\lambda)e_n, P(A-\lambda)e_n\rangle \\
&\quad + \langle P^\perp(A-\lambda)e_n, P^\perp(A-\lambda)e_n\rangle \\
&= \underbrace{\|(A-\lambda)Pe_n\|^2}_{\geq 0} + \|(A-\lambda)P^\perp e_n\|^2
\end{aligned}$$

$$\Rightarrow \|(A-\lambda)e_n\| \geq \|(A-\lambda)P^\perp e_n\|.$$

Using this in (*) implies $\|(A-\lambda)\varphi_n\| \to 0$.

Since $\|(A-\lambda)\varphi_n\| \underset{\uparrow}{=} \|(A-\lambda)P^\perp\varphi_n\|$,

use $P^\perp = (P^\perp)^2$

φ_n is a Weyl sequence of $AP^{\perp}$. Thus $\lambda \in \sigma(A\restriction_{P^{\perp}\mathcal{H}})$.

But since we assumed that λ was isolated, one also checks directly that $\lambda \notin \sigma(A\restriction_{P^{\perp}\mathcal{H}})$ by def. of the eigenspace and the projection.
(Exc.)

Contradiction. $\blacksquare$

Proposition 8.15: Let A, B self-adjoint, with $D(A) = D(B)$. Let $(A-B)$ be relatively compact w.r.t. A or B. Then

$$\sigma_{ess}(A) = \sigma_{ess}(B).$$

Proof: $(A+i)(B+i)^{-1}$ and $(B+i)(A+i)^{-1}$ are well-defined and, since self-adjoint operators are closed, also closed.
Thus they are bounded by the closed graph theorem.

If $(A-B)(A+i)^{-1}$ is compact, then

$$(A-B)(B+i)^{-1} = \underbrace{(A-B)(A+i)^{-1}}_{\text{compact}}\underbrace{(A+i)(B+i)^{-1}}_{\text{bounded}}$$

by Prop. 8.4 (c)

is compact.

Let $\lambda \in \sigma_{ess}(A)$ and $(e_n)_n$ a corresponding Weyl sequence.

Then

$$\|(B-\lambda)e_n\| \leq \|(A-\lambda-A+B)e_n\|$$

$$\leq \|(A-\lambda)e_n\| + \|(B-A)(A+i)^{-1}(A+i)e_n\|$$

$$\longrightarrow 0$$

because $(B-A)(A+i)^{-1}$ is compact and

$$(A+i)e_n = \underbrace{(A-\lambda)e_n}_{\substack{\longrightarrow 0 \\ \text{because } e_n \text{ is} \\ \text{a Weyl sequence}}} + \underbrace{(\lambda+i)e_n}_{\longrightarrow 0}$$

$$\longrightarrow 0.$$

Thus $(e_n)_n$ is a Weyl sequence for B and value λ, thus $\lambda \in \sigma_{ess}(B)$. ▨