Theorem 8.16: If $A = A^*$, C symmetric and relatively compact w.r.t. A, then $A + C$ is selfadjoint on $D(A)$ and $\sigma_{ess}(A + C) = \sigma_{ess}(A)$.

Proof: Statement about σ_{ess} follows from Prop. 8.15. We just need to show self-adjointness, for which we use the Kato-Rellich theorem.

Recall $\|(A \pm in)\varphi\|^2 = \|A\varphi\|^2 + n^2\|\varphi\|^2$.

With $\varphi = (A \pm in)^{-1}\psi$:

$$\|\psi\|^2 = \|A(A \pm in)^{-1}\psi\|^2 + n^2\|(A \pm in)^{-1}\psi\|^2, \text{ for all } \psi \in \mathcal{H}$$

Thus $\|A(A \pm in)^{-1}\|^2 \leq 1$ and

$$\|(A \pm in)^{-1}\|^2 \leq \frac{1}{n^2}.$$

Next we show $\|C(A + in)^{-1}\| \longrightarrow 0 \ (n \to \infty)$.

By Def. of the operatornorm there exists a sequence $(\varphi_n)_n$ with $\|\varphi_n\| = 1$ such that

$$\|C(A + i\eta)^{-1}\| \leq \|C(A + i\eta)^{-1}\varphi_n\| + \frac{1}{n}.$$

Then

$$\|C(A+i\eta)^{-1}\| \leq \|C(A+i\eta)^{-1} e_n\| + \frac{1}{n}$$

$$= \| \underbrace{C(A+i)^{-1}}_{\text{compact}} \underbrace{(A+i)(A+in)^{-1} e_n}_{\text{just need to show that} \longrightarrow 0.} \| + \frac{1}{n}$$

In fact, $\forall \gamma \in D(A)$:

$$|\langle \gamma, (A+i)(A+in)^{-1} e_n \rangle|$$

$$= |\langle (A-in)^{-1}(A-i)\gamma, e_n \rangle|$$

$$\leq \underbrace{\|(A-in)^{-1}\|}_{\leq \frac{1}{n}} \, \|(A-i)\gamma\| \, \underbrace{\|e_n\|}_{=1} \longrightarrow 0.$$

Since also $\|(A+i)(A+in)^{-1} e_n\| \leq 2$, this extends from $\gamma \in D(A)$ to all $\gamma \in \mathfrak{H}$.

Thus $(A+i)(A+in)^{-1} e_n \longrightarrow 0$ as claimed.

Thus we have shown $\|C(A+in)^{-1}\| \longrightarrow 0$.

Therefore we can take n large enough such that

$$\varepsilon := \|C(A+in)^{-1}\| < 1.$$

Let $\psi := (A+in)\varphi$. Then

$$\begin{aligned}\|C\varphi\| &= \|C(A+in)^{-1}\psi\| \\ &\leq \varepsilon \|\psi\| \\ &= \varepsilon \|(A+in)\varphi\| \\ &\leq \varepsilon \|A\varphi\| + \varepsilon n \|\varphi\|.\end{aligned}$$

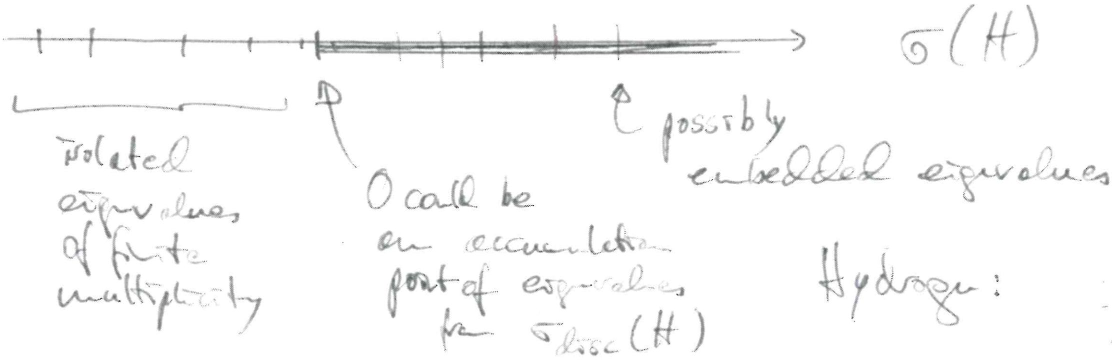
As necessary to apply Kato-Rellich.

$\blacksquare$

Example: $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$

$$\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0,\infty)$$

$$\Rightarrow \sigma(H) \supset \sigma_{ess}(H) = \sigma_{ess}(-\Delta)$$



Hydrogen:

IX MEASURABLE FUNCTIONAL CALCULUS

Let $A = A^*$ in a Hilbert space $\mathcal{H}$.
For $f \in \mathcal{S}(\mathbb{R})$ we can define $f(A) \in \mathcal{L}(\mathcal{H})$ by

$$f(A)e := \int_{\mathbb{R}} \hat{f}(t)\, e^{iAt} e \, dt \quad \forall e \in \mathcal{H},$$

for $\hat{f}(t) := (2\pi)^{-1} \int_{\mathbb{R}} f(x) e^{-itx} dx$.

Goal: Extend this to bounded Borel-measurable functions $\mathcal{B}(\mathbb{R})$.

Def. 9.1: A map $q : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ is called bounded sesquilinear form if it is linear in the second, antilinear in the first argument, and

$$\|q\| := \sup_{\substack{\|u\|=1 \\ \|v\|=1}} |q(u,v)| \quad \text{is finite.}$$

The map $u \mapsto q(u,u)$ is called the associated quadratic form.

Remarks:
- By polarization identity, the quadratic form determines the sesquilinear form
- Every $B \in \mathcal{L}(\mathcal{H})$ defines a bounded sesquilinear form by $q(u,v) := \langle u, Bv \rangle$.

The converse is also true:

Lemma 9.2: Let q a bounded sesquilinear form. Then there exists $B \in \mathcal{L}(\mathcal{H})$ such that $q(u,v) = \langle u, Bv \rangle \; \forall u,v \in \mathcal{H}$.

Proof: Exercise. $\blacksquare$

Def. 9.3: Let $\mathcal{A} \subset B(\mathbb{R}^n)$ a sub-algebra (E.g. $\mathcal{A} = C_0(\mathbb{R}^n)$). A map $\Phi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$ is called *$*$-homomorphism* if $\forall f, g \in \mathcal{A}$:

$$\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g), \; \alpha, \beta \in \mathbb{C}$$

$$\Phi(fg) = \Phi(f)\Phi(g)$$

$$\Phi(\overline{f}) = \Phi(f)^* .$$

Proposition 9.4: Let $\mathcal{A}$ one of the algebras of functions $C_0(\mathbb{R}^n)$, $C_0^\infty(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$, or $B(\mathbb{R}^n)$.
Let $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ a $*$-homomorphism.
Then for all $f \in \mathcal{A}$:

$$\| \Phi(f) \| \leq \| f \|_\infty .$$

Proof: Note that for every self-adjoint operator

$$B \geq 0 \implies \|B\| = \sup_{\|e\|=1} \langle e, Be \rangle \quad (1)$$

and

$$\|B^* B\| = \|B\|^2. \quad (2)$$

Since $\Phi(\alpha f) = \alpha \Phi(f)$ $\forall \alpha \in \mathbb{C}$ it is sufficient to show $\|\Phi(f)\| \leq 1$, $\forall f \in \mathcal{A}$ with $\|f\|_\infty < 1$.

So let $f \in \mathcal{A}$ with $\|f\|_\infty < 1$.

Then

$$g := f\sqrt{1-|f|^2} \in \mathcal{A}$$

(for the case of $f \in \mathcal{S}(\mathbb{R}^n)$ remember to check also the derivatives).

We have

$$0 \leq \Phi(g)^* \Phi(g) = \Phi(|g|^2)$$

$$= \Phi(|f|^2 - |f|^4) = \Phi(|f|^2) - \Phi(|f|^4). \quad (3)$$

Since $\Phi(|f|^2) = \Phi(f^*)\Phi(f) \geq 0$

and $\Phi(|f|^4) \geq 0$,

we conclude by (1) and (3) that

$$\|\Phi(|f|^4)\| \leq \|\Phi(|f|^2)\|. \quad (4)$$

By (2) we also have

$$\|\Phi(|f|^2)\| = \|\Phi(f)^*\Phi(f)\| = \|\Phi(f)\|^2$$

$$\|\Phi(|f|^4)\| = \|\Phi(f)\|^4.$$

Thus (4) is equivalent to

$$\|\Phi(f)\|^4 \leq \|\Phi(f)\|^2$$

This implies $\|\Phi(f)\| \leq 1$. ∎

Proposition 9.5: Every $*$-homomorphism $\Phi : S(\mathbb{R}^n) \longrightarrow \mathcal{L}(\mathcal{H})$ has a unique extension to a $*$-homomorphism $\tilde{\Phi} : B(\mathbb{R}^n) \longrightarrow \mathcal{L}(\mathcal{H})$ with the property:

$$(*) \quad \left.\begin{array}{l} f_k \in B(\mathbb{R}^n) \text{ and} \\ \sup_k \|f_k\|_\infty < \infty \text{ and} \\ f_k(x) \to f(x) \ \forall x \in \mathbb{R}^n \end{array}\right\} \Rightarrow \tilde{\Phi}(f) = \text{s-}\lim_{k\to\infty} \tilde{\Phi}(f_k).$$

Remark: To abbreviate, we write $f_k \xrightarrow{p} f$ if $f_k(x) \longrightarrow f(x) \ \forall x \in \mathbb{R}^n$ and $\sup_k \|f_k\|_\infty$.

To prove 9.5 we need 2 results from measure theory.

Def. 9.6: Let $\mathcal{B}(\mathbb{R}^n)$ the set of bounded Borel measurable functions $\mathbb{R}^n \to \mathbb{C}$.
For $f \in \mathcal{B}(\mathbb{R}^n)$ set

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^n} |f(x)| < +\infty.$$

(We do not identify functions that differ only on a null set.)

Proposition 9.7: Let $\mathcal{F} \subset \mathcal{B}(\mathbb{R}^n)$ satisfy:

(a) $C_0(\mathbb{R}^n) \subset \mathcal{F}$

(b) $\mathcal{F}$ is closed w.r.t. "$\xrightarrow{p}$", i.e., $f_k \in \mathcal{F}$ and $f_k \xrightarrow{p} f$ implies $f \in \mathcal{F}$.

Then $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$.

Sketch of proof of Prop. 9.7: Let $\mathcal{D} := \bigcap\limits_{\substack{\mathcal{F} \text{ satisfies} \\ (a) \text{ and } (b)}} \mathcal{F}$.

Show: (1) $f, g \in \mathcal{D} \Rightarrow f \cdot g \in \mathcal{D}$

(2) $f, g \in \mathcal{D} \Rightarrow \alpha f + \beta g \in \mathcal{D}$, $\alpha, \beta \in \mathbb{C}$.

(3) $C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset \mathcal{D}$.

(4) $\chi_M \in \mathcal{D}$ if $M \subset \mathbb{R}^n$ is closed.

(5) $\Sigma := \{M \subset \mathbb{R}^n \mid \chi_M \in \mathcal{D}\}$ is a σ-algebra.

$\Rightarrow$ $\chi_M \in \mathcal{D}$ for all Borel-measurable $M \subset \mathbb{R}^n$.

$\Rightarrow$ $\mathcal{D}$ contains all Borel-measurable simple functions $\sum_{i=1}^N \alpha_i \chi_{M_i}$, $\alpha_i \in \mathbb{C}$.

$\Rightarrow$ $\mathcal{D}$ contains $\mathcal{B}(\mathbb{R}^n)$.

$\blacksquare$

Def. 9.8: Let μ be a Borel measure on $\mathbb{R}^n$ with $\mu(K) < \infty$ for all compact sets $K \subset \mathbb{R}^n$.

Then
$$L : C_0(\mathbb{R}^n) \longrightarrow \mathbb{C}$$
$$f \longmapsto \int_{\mathbb{R}^n} f \, d\mu$$
is a positive linear functional, that is L is linear and $f \geq 0 \Rightarrow L(f) \geq 0$.

Theorem 9.9: (Riesz-Markov)

Let $L : C_0(\mathbb{R}^n) \longrightarrow \mathbb{C}$ a positive linear functional, then there exists a Borel measure μ with
$$L(f) = \int_{\mathbb{R}^n} f \, d\mu \quad \forall f \in C_0(\mathbb{R}^n).$$
If $K \subset \mathbb{R}^n$ is compact, then $\mu(K) < +\infty$.

Proof: Reed & Simon, Vol. I, Thm. IV.18 or Rudin, Real & Complex Analysis, Thm. 2.14 + 2.18. ∎

Remark:

The Riesz-Markov theorem is not surprising. Given $M \subset \mathbb{R}^n$, we can take the characteristic function $\chi_M : \mathbb{R}^n \longrightarrow \mathbb{R}$ and would like to define $\mu(M) := L(\chi_M)$. We "just" have to overcome the problem that χ_M is <u>not</u> continuous!