## Proof of Prop. 9.5:

1) Uniqueness: Let $\Phi_1, \Phi_2 : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ two $*$-homomorphisms with $\Phi_{1,2}\restriction_{\mathcal{S}(\mathbb{R}^n)} = \Phi$ and satisfying $(*)$.

From Prop. 9.4 we have
$$\|\Phi_i(f_1 - f_2)\| \leq \|f_1 - f_2\|_\infty,$$
i.e. $\Phi_i$ are continuous.

Thus $\Phi_1 = \Phi_2$ on $\overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_\infty}$.

Let $\mathcal{F} := \{ f \in \mathcal{B}(\mathbb{R}^n) : \Phi_1(f) = \Phi_2(f) \}$.

It is well-known that $C_0(\mathbb{R}^n) \subset \overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_\infty}$.

Thus $C_0(\mathbb{R}^n) \subset \mathcal{F}$.

Let $(f_n)_n$ a sequence in $\mathcal{F}$ with $f_n \xrightarrow{p} f$. Then by assumption
$$\Phi_1(f) = s\text{-}\lim_{n\to\infty} \Phi_1(f_n) = s\text{-}\lim_{n\to\infty} \Phi_2(f_n) = \Phi_2(f).$$
Thus $f \in \mathcal{F}$.

Thus (a) + (b) of Prop. 9.7 hold, $\Rightarrow \mathcal{F} = \mathcal{B}(\mathbb{R}^n)$.

(2) Existence: Recall that by Prp. 8.4, $\Phi$ is continuous, so that its extension to $\overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_\infty}$ exists.

We denote this extension also by $\Phi$.

For $f \in C_0(\mathbb{R}^n)$, $f \geq 0$, and all $\varphi \in \mathcal{H}$:

$$\langle \varphi, \Phi(f)\varphi\rangle = \|\Phi(\sqrt{f})\varphi\|^2 \geq 0.$$

Thus, for every $\varphi \in \mathcal{H}$,

$$L_\varphi : C_0(\mathbb{R}^n) \longrightarrow \mathbb{C}$$
$$f \longmapsto \langle \varphi, \Phi(f)\varphi\rangle$$

is a positive linear functional.

By Riesz-Markov there exists a Borel-measure $\mu_\varphi$ such that

$$\langle \varphi, \Phi(f)\varphi\rangle = \int_{\mathbb{R}^n} f \, d\mu_\varphi \quad \forall f \in C_0(\mathbb{R}^n).$$

(1) We show $\mu_\varphi(\mathbb{R}^n) \leq \|\varphi\|^2$:

Let $(f_k)_k$ a sequence in $C_0(\mathbb{R}^n, [0,1])$ with $f_k \nearrow 1$. By monotone convergence

$$\mu_\varphi(\mathbb{R}^n) = \int_{\mathbb{R}^n} d\mu_\varphi = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k \, d\mu_\varphi$$
$$= \lim_{k\to\infty} \langle \varphi, \Phi(f_k)\varphi\rangle \leq \langle \varphi, \varphi\rangle$$

because

$$\langle \varphi, \Phi(f_k)\varphi\rangle \leq \|\varphi\| \, \|\Phi(f_k)\varphi\|$$

$$\leq \|\varphi\| \, \|f_n\|_\infty \|\varphi\| \leq \|\varphi\|^2.$$

(2) We show that $\varphi \mapsto \int f \, d\mu_\varphi$ is a quadratic form for all $f \in \mathcal{B}(\mathbb{R}^n)$:

By (1) all functions $f \in \mathcal{B}(\mathbb{R}^n)$ are integrable w.r.t. $\mu_\varphi$.
So set $I_\varphi(f) := \int f \, d\mu_\varphi$.
For $\varphi, \psi \in \mathcal{H}$, set

$$q_f(\varphi, \psi) := \frac{1}{4}\left(I_{\varphi+\psi}(f) - I_{\varphi-\psi}(f)\right) + \frac{1}{4i}\left(I_{\varphi+i\psi}(f) - I_{\varphi-i\psi}(f)\right).$$

We have $C_0(\mathbb{R}^n) \subset \mathcal{F}$ because

$$q_f(\varphi, \psi) = \frac{1}{4}\left(I_{\varphi+\psi}(f) - I_{\dots}\right)$$
$$= \frac{1}{4}\left(L_{\varphi+\psi}(f) - L_{\dots}\right)$$

and by the polarization identity

$$= \langle \varphi, \Phi(f) \psi \rangle.$$

Let $\mathcal{F} \subset \mathcal{B}(\mathbb{R}^n)$ the set of functions $f$ where $q_f$ is a bounded sesquilinear form.

By (1) and dominated convergence

$$f_k \xrightarrow{p} f \Rightarrow q_{f_k}(\varphi, \psi) = q_f(\varphi, \psi).$$

Thus (a) + (b) of Prp. 9.7 hold and $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$.

By Lemma 9.2, every bounded sesquilinear form can be represented by an operator:

$$\forall f \in \mathcal{B}(\mathbb{R}^n) \quad \exists \tilde{\Phi}(f) \in \mathcal{L}(\mathcal{H}):$$

$$q_f(\varphi, \psi) = \langle \varphi, \tilde{\Phi}(f)\psi \rangle.$$

(3) We show that $f \mapsto \tilde{\Phi}(f)$ is a $*$-homomorphism.

Linearity is easy. Multiplicativity:

Let $g \in C_0(\mathbb{R}^n)$ and $\mathcal{F}_g := \{ f \in \mathcal{B}(\mathbb{R}^n) : \tilde{\Phi}(g)\tilde{\Phi}(f) = \tilde{\Phi}(gf) \}$

Then $C_0(\mathbb{R}^n) \subset \mathcal{F}_g$ and

$f_n \in \mathcal{F}_g$, $f_n \xrightarrow{p} f$, implies using dominated convergence

$$\begin{array}{ccc} \langle \varphi, \hat{\Phi}(g)\hat{\Phi}(f_n)\psi \rangle & = & \langle \varphi, \hat{\Phi}(g f_n)\psi \rangle \\ \downarrow & & \downarrow \\ \langle \varphi, \hat{\Phi}(g)\hat{\Phi}(f)\psi \rangle & & \langle \varphi, \hat{\Phi}(gf)\psi \rangle \end{array}$$

So $\mathcal{F}_g = \mathcal{B}(\mathbb{R}^n)$.

If $g \in \mathcal{B}(\mathbb{R}^n)$ (instead of $C_0(\mathbb{R}^n)$):

$C_0(\mathbb{R}^n) \subset \mathcal{F}_g$ by the previous step,

closure under $f_n \xrightarrow{p} f$ as before.

Thus $\mathcal{F}_g = \mathcal{B}(\mathbb{R}^n)$.

$\tilde{\Phi}(\bar{f}) = \tilde{\Phi}(f)^*$ similarly.

(4) We show $f_k \xrightarrow{p} f \Rightarrow \tilde{\Phi}(f) = \text{s-}\lim_{k\to\infty} \tilde{\Phi}(f_k)$.

$f_k \xrightarrow{p} f$ implies $\langle \varphi, \tilde{\Phi}(f_k)\psi\rangle \to \langle \varphi, \tilde{\Phi}(f)\psi\rangle$

for all $\varphi, \psi \in \mathcal{H}$.

(*) Thus $\tilde{\Phi}(f_k)\psi \longrightarrow \tilde{\Phi}(f)\psi$ (weakly).

In particular:

$$\|\tilde{\Phi}(f_k)\psi\|^2 = \langle \psi, \tilde{\Phi}(|f_k|^2)\psi\rangle$$

$$\longrightarrow \langle \psi, \tilde{\Phi}(|f|^2)\psi\rangle$$

(**)

$$= \|\tilde{\Phi}(f)\psi\|^2.$$

In general, $\varphi_k \rightarrow \varphi$ and $\|\varphi_k\| \rightarrow \|\varphi\|$

implies

$$\|\varphi_k - \varphi\|^2 = \langle \varphi_k, \varphi_k\rangle + \langle \varphi, \varphi\rangle - 2\,\mathrm{Re}\langle \varphi_k, \varphi\rangle$$

$$= \|\varphi_k\|^2 + \|\varphi\|^2 - 2\,\mathrm{Re}\langle \varphi_k, \varphi\rangle$$

$$\longrightarrow \|\varphi\|^2 + \|\varphi\|^2 - 2\|\varphi\|^2$$

$$= 0.$$

Applied with (*) & (**): $\tilde{\Phi}(f_k)\psi \longrightarrow \tilde{\Phi}(f)\psi$.

$\blacksquare$

Theorem 9.10: (existence of the measurable functional calculus)

Let $U : \mathbb{R}^n \to \mathcal{L}(\mathcal{H})$ a $n$-parameter SCUG. Then there exists a unique $\tilde{\Phi} : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ with:

(a) $U(t) = \tilde{\Phi}(e_t) \quad \forall t \in \mathbb{R}^n, \; e_t(x) := e^{-it\cdot x}$.

(b) $\tilde{\Phi}$ is a $*$-homomorphism.

(c) $f_n \xrightarrow{p} f \Rightarrow \tilde{\Phi}(f) = \operatorname{s-lim}_{n\to\infty} \tilde{\Phi}(f_n)$.

Moreover, for all $f \in \mathcal{B}(\mathbb{R}^n)$:

- $\|\tilde{\Phi}(f)\| \leq \|f\|_\infty$
- $f \geq 0 \Rightarrow \tilde{\Phi}(f) \geq 0$.

Proof: $\|\tilde{\Phi}(f)\| \leq \|f\|_\infty$ follows from (b) and Prp. 9.4.

$f \geq 0 \Rightarrow \tilde{\Phi}(f) = \tilde{\Phi}(\sqrt{f})^* \tilde{\Phi}(\sqrt{f}) \geq 0$.

Remains to show uniqueness and existence.

Uniqueness: Let $f \in \mathcal{S}(\mathbb{R}^n)$, by Fourier transform $f(x) = \int \hat{f}(t) e^{it\cdot x}\, dt$

Then there exists a sequence of function, given by the Riemann sums

$$f_N(x) = \sum_{k \in \Lambda_N} \hat{f}(t_k)\, e^{ix\cdot t_k} \Delta t_k ,$$

with $f_N(x) \longrightarrow f(x)\ \forall x \in \mathbb{R}^n$.

We have $\|f_N\|_\infty \le \|\hat{f}\|_1 + 1$.

So for all $\varphi, \psi \in \mathfrak{H}$:

$$\langle \varphi, \tilde{\Phi}(f)\psi \rangle \overset{(c)}{=} \lim_{N\to\infty} \langle \varphi, \tilde{\Phi}(f_N)\psi \rangle$$

$$\overset{(a),(b)}{=} \lim_{N\to\infty} \sum_{k\in\Lambda_N} \hat{f}(t_k) \langle \varphi, U(-t_k)\psi \rangle \Delta t_k$$

$$= \int_{\mathbb{R}^n} \hat{f}(t) \langle \varphi, U(-t)\psi \rangle \, dt .$$

So $\tilde{\Phi}\restriction_{\mathcal{S}(\mathbb{R}^n)}$ is uniquely determined by $U$.

Uniqueness of $\tilde{\Phi}$ on $\mathcal{B}(\mathbb{R}^n)$ follows by Prop. 9.5

Existence: Given $f \in \mathcal{S}(\mathbb{R}^n)$, define $\Phi(f) \in \mathcal{L}(\mathcal{H})$ as the operator representing the bounded sesquilinear form

$$\langle \varphi, \Phi(f)\psi \rangle := \int \hat{f}(t) \langle \varphi, U(-t)\psi\rangle dt$$

It is easy to see that $f \mapsto \Phi(f)$ is linear and $\Phi(\bar{f}) = \Phi(f)^*$.

For multiplicativity, let $f, g \in \mathcal{S}(\mathbb{R}^n)$: then

$$f(x) g(x) = \int dt \int ds\, \hat{f}(t)\, \hat{g}(s)\, e^{i(t-s)x}$$

$$= \int dr \left( \int ds\, \hat{f}(r-s) \hat{g}(s) \right) e^{i r \cdot x}.$$

Thus

$$\Phi(fg) = \int dr \left( \int ds\, \hat{f}(r-s) \hat{g}(s) \right) U(-r)$$

$$= \int dt \left( \int ds\, \hat{f}(t)\, \hat{g}(s) \right) U(-t)\, U(-s)$$

$$= \Phi(f)\, \Phi(g).$$

So $\Phi$ is a $*$-homomorphism on $\mathcal{S}(\mathbb{R}^n)$, and by Prop. 8.5 there exists a unique extension $\tilde{\Phi}$ to $\mathcal{B}(\mathbb{R}^n)$ satisfying (b), (c).

Remains to show (a), $\tilde{\Phi}(e_t) = U(t)$.

(1) We show that $M := \{\Phi(f)\varphi : f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

In fact, assume $M$ was not dense in $\mathcal{H}$.
Then there exists $\varphi \neq 0$ with $\varphi \perp M$.
In particular $\langle \varphi, \tilde{\Phi}(f)\varphi \rangle = 0 \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$
Then $\int \hat{f}(t) \langle \varphi, U(-t)\varphi \rangle \, dt = 0$,
for all $\hat{f} \in C_0^\infty$.
Then $\langle \varphi, U(-t)\varphi \rangle = 0 \quad \forall t \in \mathbb{R}^n$.
In particular, for $t = 0$, $\langle \varphi, \varphi \rangle = 0$, ↯.

(2) We show $\tilde{\Phi}(e_t)\tilde{\Phi}(f)\varphi = U(t)\tilde{\Phi}(f)\varphi$
for all $f \in \mathcal{S}(\mathbb{R}^n)$, all $\varphi \in \mathcal{H}$.

In fact, $e_t(x) f(x) = \int \hat{f}(s) e^{i(s-t)\cdot x} \, ds$

$$= \int \hat{f}(s+t) e^{is\cdot x} \, ds,$$

thus

$$\tilde{\Phi}(e_t)\tilde{\Phi}(f) = \tilde{\Phi}(e_t f) = \Phi(e_t f)$$

$$= \int \hat{f}(s+t) U(-s) \, ds = \int \hat{f}(s) U(-s) U(t) \, ds$$

$$= U(t) \hat{\Phi}(f).$$

∎