Def. 8.11: A map $\phi : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ with

(a) $\phi(1) = \mathbb{1}$ $\quad (1(x) = 1 \ \forall x \in \mathbb{R}^n)$

(b) ϕ is a $*$-homomorphism.

(c) $f_n \xrightarrow{p} f \Rightarrow \phi = \underset{n\to\infty}{s\text{-}\lim}\, \phi(f_n)$

is called a measurable functional calculus.

Remark: By theorem 8.10, for every SCUG there exists a unique measurable functional calculus ϕ with $U(t) = \phi(e_t)$. $(*)$

Conversely, by $(*)$ every measurable functional calculus defines a SCUG.

Notation: If $U(t) = e^{-iAt}$, we write

$$\phi(f) =: f(A).$$

A is an operator, $f(A)$ is an operator.

That means, we can take functions of operators now.

But up to now only bounded functions.

Next step: f not bounded, only measurable.

X The Spectral theorem:

Prop. 10.1: Let $\Phi : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ a measurable functional calculus.

Let $P_\Omega := \Phi(\chi_\Omega)$ for $\Omega \subset \mathbb{R}^n$ Borel-measurable

Then:

(i) P_Ω is an orthogonal projection ($P_\Omega^2 = P_\Omega$, $P_\Omega^* = P_\Omega$)

(ii) $P_\emptyset = 0$, $P_{\mathbb{R}^n} = \mathbb{1}$

(iii) $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ and Ω_i pairwise disjoint, then $P_\Omega = \operatorname{s-lim}_{N \to \infty} \sum_{i=1}^{N} P_{\Omega_i}$.

(iv) $P_{\Omega_i} P_{\Omega_k} = P_{\Omega_i \cap \Omega_k}$.

Proof: Exercise. ∎

Def. 10.2: A map $P: \begin{pmatrix} \text{Borel measurable} \\ \text{sets} \end{pmatrix} \to \mathcal{L}(\mathcal{H})$ satisfying (i)–(iv) from Prop. 10.1 is called projection (operator) valued measure (POVM, PVM).

For all $\varphi \in \mathcal{H}$, $\mu_\varphi(\Omega) := \langle \varphi, P_\Omega \varphi \rangle$ defines a Borel measure on $\mathbb{R}^n$.

Corollary 10.3: A PVM is monotone and subadditive:

$$\Omega_1 \subset \Omega_2 \Rightarrow P_{\Omega_1} \leq P_{\Omega_2},$$

$$\Omega = \bigcup_{k=1}^{n} \Omega_k \Rightarrow P_\Omega \leq \sum_{k=1}^{n} P_{\Omega_k}.$$

If $\|\varphi\| = 1$, then $\mu_\varphi(\mathbb{R}^n) = 1$, so μ_φ is a probability measure.

Def. 10.3: The support $\operatorname{supp}(P) \subset \mathbb{R}^n$, is defined by

$$x \in \operatorname{supp} P \Leftrightarrow P_\Omega \neq 0 \text{ for every open neighbourhood } \Omega \ni x.$$

Lemma 10.4: Let $\Phi : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ a measurable functional calculus and $P_\Omega := \Phi(\chi_\Omega)$. Then:

(i) $S := \operatorname{supp}(P)$ is closed.

(ii) $P_S = \mathbb{1}$, $P_{\mathbb{R}^n \setminus S} = 0$

(iii) If $f \in \mathcal{B}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$:

$$\|\Phi(f)\| = \sup_{x \in S} |f(x)|.$$

Proof: (i) obvious from Def. 10.3.

(ii) We show $P_{\mathbb{R}^n \setminus S} = 0$.

Let $K \subset \mathbb{R}^n \setminus S$ compact.

Then there exist open sets $\Omega_1, \dots, \Omega_N$ with $P_{\Omega_i} = 0$ and $K \subset \bigcup_{i=1}^N \Omega_i$.

Thus $P_K \leq \sum_{i=1}^N P_{\Omega_i} = 0 \Rightarrow P_K = 0$.

Let $K_m := \{x \in \mathbb{R}^n : |x| \leq m$ and $\operatorname{dist}(x, S) \geq 1/m\} \subset \mathbb{R}^n \setminus S$.

K_m is compact, so $P_{K_m} = 0$.

Moreover $\bigcup_{m \geq 1} K_m = \mathbb{R}^n \setminus S$.

Thus $P_{\mathbb{R}^n \setminus S} \leq \sum_{m=1}^{\infty} P_{K_m} = 0.$

(iii) By (ii): $\Phi(f) = \Phi(\chi_S f).$

Thus $\|\Phi(f)\| \leq \|\chi_S f\|_\infty.$

Let $x \in S$. Sufficient to show $\|\Phi(f)\| \geq |f(x)|$

For $f(x) = 0$ trivial.

So assume $f(x) \neq 0$.

Since f is continuous, for every $\varepsilon > 0$ there exists an open neighbourhood $U \ni x$ with

$$|f(y)| > |f(x)| - \varepsilon \geq 0 \quad \forall y \in U.$$

By definition of S: $P_U \neq 0$.

Thus exists $e \in \mathcal{H}$ with $\|e\| = 1$ and $P_U e = e$.

Then

$$\begin{aligned}\|\Phi(f)e\|^2 &= \langle e, \Phi(|f|^2) e\rangle \\ &= \langle e, \Phi(|f|^2 \chi_U) e\rangle \\ &\geq (|f(x)| - \varepsilon)^2 \langle e, P_U e\rangle \\ &= (|f(x)| - \varepsilon)^2, \quad \forall \varepsilon > 0.\end{aligned}$$

$\Rightarrow \|\Phi(f)\| \geq |f(x)|.$ ∎

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Def. 10.5: Let $\Phi : \mathcal{B}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ a measurable functional calculus and $f : \mathbb{R}^n \to \mathbb{C}$ Borel measurable (possibly unbounded!).

Define an operator $\Phi(f)$ by:

$$D(\Phi(f)) := \Phi\left(\frac{1}{1+|f|}\right)\mathcal{H}$$

$$\Phi(f)\varphi := \Phi\left(\frac{f}{1+|f|}\right)\gamma$$

when $\varphi = \Phi\left(\frac{1}{1+|f|}\right)\gamma$.

Lemma 10.6: With the above definitions:

(i) $\Phi(f)$ is well-defined and densely defined

(ii) $D(\Phi(f)) = \{\varphi \in \mathcal{H} : \int |f|^2 \, d\mu_\varphi < +\infty\}$

and for $\varphi \in D(\Phi(f))$:

$$\|\Phi(f)\varphi\|^2 = \int |f|^2 \, d\mu_\varphi$$

$$\langle \varphi, \Phi(f)\varphi \rangle = \int f \, d\mu_\varphi$$

where $\mu_\varphi(\Omega) := \langle \varphi, \Phi(\chi_\Omega)\varphi \rangle$

(iii) Let $f_n \in \mathcal{B}(\mathbb{R}^n)$, $|f_n| \leq |f|$, $f_n(x) \to f(x)$ $\forall x \in \mathbb{R}^n$. Then for all $\varphi \in D(\Phi(f))$:

$$\Phi(f)\varphi = \lim_{n \to \infty} \Phi(f_n)\varphi.$$

(iv) If $g, fg \in \mathcal{B}(\mathbb{R}^n)$, then
$\Phi(g)\mathcal{H} \subset D(\Phi(f))$ and
$$\Phi(f)\Phi(g) = \Phi(fg).$$

(v) If $\overline{f} = f$, then $\Phi(f) = \Phi(f)^*$
and for $z \in \mathbb{C} \setminus \mathbb{R}$:
$$(\Phi(f) - z)^{-1} = \Phi\left(\frac{1}{f-z}\right).$$

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Proof: (i) Well-defined: for $\Phi(f)\varphi$ to be uniquely defined, we need $\Phi\left(\frac{1}{1+|f|}\right)$ to be injective.

Let $\Phi\left(\frac{1}{1+|f|}\right)\psi = 0$.

Through approximation by elementary functions one shows

$$\langle \varphi, \Phi(g)\varphi\rangle = \int g \, d\mu_\varphi \quad \text{for } g \in \mathcal{B}(\mathbb{R}^n).$$

Thus

$$0 = \langle \psi, \Phi\left(\frac{1}{1+|f|}\right)\psi\rangle = \int \frac{1}{1+|f|} \, d\mu_\psi .$$

Since $\frac{1}{1+|f(x)|} > 0$ for all $x \in \mathbb{R}^n$,

this implies $0 = \mu_\psi(\mathbb{R}^n) = \|\psi\|^2$.

$$\Rightarrow \psi = 0.$$

Densely defined: Let $\varphi \in \mathcal{H}$, χ_k the characteristic function of $\{x \in \mathbb{R}^n : |f(x)| \leq k\}$.

Then

$$\Phi(\chi_k)\varphi = \Phi\left(\frac{1}{1+|f|}\right)\Phi(\chi_k(1+|f|))\varphi \in D(\Phi(f)).$$

By $\chi_k \xrightarrow{p} 1$ we get $\Phi(\chi_k)\varphi \to \varphi$.

Thus $D(\Phi(f))$ dense in $\mathcal{H}$.

(iii) Let $\varphi = \Phi\left(\frac{1}{1+|f|}\right)\gamma$. Then

$$\Phi(f_n)\varphi = \Phi\left(\frac{f_n}{1+|f|}\right)\gamma \longrightarrow \Phi\left(\frac{f}{1+|f|}\right)\gamma$$
$$= \Phi(f)\varphi$$

because $|f_n| \leq |f|$

$$\Rightarrow \left|\frac{f_n}{1+|f|}\right| \leq \frac{|f|}{1+|f|} \leq 1 \Rightarrow \frac{f_n}{1+|f|} \xrightarrow{p} \frac{f}{1+|f|}.$$

(ii) Let $\varphi \in D(\Phi(f))$, $f_n := \chi_n f$. By (iii):

$$\|\Phi(f)\varphi\|^2 = \lim_{n\to\infty} \|\Phi(f_n)\varphi\|^2$$
$$= \lim_{n\to\infty} \int |f_n|^2 d\mu_\varphi = \int |f|^2 d\mu_\varphi$$

by dominated convergence.

Moreover

$$\langle \varphi, \Phi(f)\varphi\rangle = \lim_{n\to\infty} \langle \varphi, \Phi(f_n)\varphi\rangle$$
$$= \lim_{n\to\infty} \int f_n \, d\mu_\varphi = \int f \, d\mu_\varphi,$$

since $|f_n| \leq |f| \in L^2(\mu_\varphi) \subset L^1(\mu_\varphi)$.

because μ_φ is a bounded measure.

Domain: exercise.

(iv) Let $\varphi \in \mathfrak{A}$. Then

$$\Phi(g)\varphi = \Phi\left(\frac{1}{1+|f|} g(1+|f|)\right)\varphi$$
$$= \Phi\left(\frac{1}{1+|f|}\right)\Phi\big(\underbrace{g(1+|f|)}_{\in \mathcal{B}(\mathbb{R}^n) \text{ by assumption}}\big)\varphi$$