Thus $\Phi(g)\varphi \in D(\Phi(f))$

and $\Phi(f)\Phi(g)\varphi$

$$= \Phi\left(\frac{f}{1+|f|}\right)\Phi((1+|f|)g)\varphi = \Phi(fg)\varphi.$$

(v) Let $f = \overline{f}$, $f_k := \chi_k f$ ("χ_k as above, char. fct. of where $|f| \leq k$")

Then for all $\psi, \varphi \in D(\Phi(f))$:

$$\langle \varphi, \Phi(f)\psi \rangle = \lim_{k\to\infty} \langle \varphi, \Phi(f_k)\psi \rangle$$

$$= \lim_{k\to\infty} \langle \Phi(f_k)^* \varphi, \psi \rangle$$

$$= \lim_{k\to\infty} \langle \Phi(f_k)\varphi, \psi \rangle = \langle \Phi(f)\varphi, \psi \rangle.$$

So $\Phi(f) \subset \Phi(f)^*$.

Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then $(f-z)^{-1} \in \mathcal{B}(\mathbb{R})$.

By (iv):

$$(\Phi(f)-z)\Phi\left(\frac{1}{f-z}\right) = \Phi\left(\frac{f-z}{f-z}\right) = \Phi(1) = \mathbb{1}.$$

Thus $\mathrm{Ran}(\Phi(f)-z) = \mathcal{H}$.

The basic criterion of self-adjointness (1st week of lectures) implies

$\Phi(f) = \Phi(f)^*$. ▨

Theorem 10.7 (Spectral theorem)

Let $\Phi : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ a measurable functional calculus, let $I : \mathbb{R} \longrightarrow \mathbb{R}$, $I(x) = x$. Then $A := \Phi(I)$ is a selfadjoint operator.

Conversely, for every $A = A^*$ there is a measurable functional calculus Φ uniquely determined by $A = \Phi(I)$.

$\sigma(A)$ is the support of the PVM $\Omega \mapsto \Phi(\chi_\Omega)$.

Proof: $\Phi(I) = \Phi(I)^*$ follows from 13.3.

If $A = A^*$, we define Φ using Thm. 9.10 through $e^{-iAt} = \Phi(e_t)$, $e_t = e^{-itx}$.

In particular, A generates the SCUG $\Phi(e_t)$.

By $\left|\frac{e^{-itx} - 1}{t}\right| \leq |x|$, $i\,\frac{e^{-itx} - 1}{t} \to x \ (t \to 0)$,

and lemma 10.6 (iii) follows that for $\varphi \in D(\Phi(I))$ we have

$$\Phi(I)\varphi = \lim_{t \to 0} i\,\Phi\left(\frac{e_t - 1}{t}\right)\varphi = A\varphi.$$

Thus $\Phi(I) \subset A$.

Since $A = A^*$ and $\Phi(I) = \Phi(I)^*$:

$$A^* \subset \Phi(I)^* \implies \Phi(I) = A.$$

For every measurable functional calculus $\Phi(I)$ generates $\Phi(e_t)$. Thus $\Phi(e_t)$ and by Thm. 9.10 also Φ are uniquely specified through $\Phi(I)$.

By lemmas 10.4 and 10.6 we have, for $z \in \mathbb{C} \setminus \mathbb{R}$:

$$\|(z-A)^{-1}\| = \|(z-\Phi(I))^{-1}\|$$
$$= \left\|\Phi\left(\frac{1}{z-I}\right)\right\| = \sup_{x \in S} \frac{1}{|z-x|}$$
$$= \frac{1}{\operatorname{dist}(z,S)}.$$

("S the support of the corresponding PVM.")

We conclude that $\sigma(A) = S$:

Let $(z_n)_n$ a sequence in $\mathbb{C} \setminus \mathbb{R}$ such that $z_n \to \lambda \in \mathbb{R}$.

If $\lambda \in \sigma(A)$, then $\|(z_n - A)^{-1}\| \to \infty$.

Thus $\operatorname{dist}(z_n, S) \to 0$, thus $\lambda \in \overline{S}$, but S is always closed.

Conversely, if $\lambda \in S$, then $\operatorname{dist}(z_n, S) \to 0$.

So $\|(z_n - A)^{-1}\| \to \infty$. Thus $\lambda \notin \mathcal{S}(A)$. $\blacksquare$

Remarks: (i) The identity $A = \Phi(I)$ in the form

$$\langle \varphi, A\varphi \rangle = \int_{\sigma(A)} \lambda \, d\mu_\varphi(\lambda)$$

is formally written as $A = \int_{\sigma(A)} \lambda \, dP(\lambda)$.

(P the PVM).

This is the spectral decomposition, generalizing linear algebra:

$$A = \sum_{i=1}^{n} \lambda_i P_i \quad \text{on } \mathcal{H} = \mathbb{C}^n,$$

where λ_i are the eigenvalues of A and P_i the projections on the corresponding eigenspace.

(ii) A is bounded if and only if $\sigma(A)$ is a bounded set. Moreover

$$\|A\| = \sup_{\lambda \in \sigma(A)} |\lambda| .$$

Proof: Let $\sigma(A)$ bounded. Then for $\varphi \in D(A)$.

$$\|A\varphi\|^2 = \int_{\sigma(A)} |\lambda|^2 \, d\mu_\varphi \leq \sup_{\lambda \in \sigma(A)} |\lambda|^2 \, \|\varphi\|^2 .$$

Thus A is bounded. $\square$

(iii) Spectral subspaces: Let $\Omega \subset \mathbb{R}$ a Borel set, $P_\Omega := \Phi(\chi_\Omega)$, $\mathcal{H}_\Omega := P_\Omega \mathcal{H}$.

Then $P_\Omega A \subset A P_\Omega$ and

$$A\restriction_{\mathcal{H}_\Omega} : P_\Omega D(A) \subset \mathcal{H}_\Omega \to \mathcal{H}_\Omega$$

is self-adjoint. Exc.

If Ω is open, then

$$\sigma(A) \cap \Omega \subset \sigma(A\restriction_{\mathcal{H}_\Omega}) \subset \sigma(A) \cap \overline{\Omega}.$$

Proof: • For $\lambda \in \sigma(A) \cap \Omega$ and $\varepsilon > 0$ small enough: $(\lambda-\varepsilon, \lambda+\varepsilon) \subset \Omega$

and $P_{(\lambda-\varepsilon, \lambda+\varepsilon)} \neq 0$ since $\sigma(A)$ and Ω are both open.

So $\lambda \in \sigma(A\restriction_{\mathcal{H}_\Omega})$.

• For $\lambda \notin \sigma(A) \cap \overline{\Omega}$: $\delta := \operatorname{dist}(\lambda, \sigma(A) \cap \overline{\Omega}) > 0$,

thus for all $\varphi \in P_\Omega D(A)$:

$$\|(A-\lambda)\varphi\|^2 = \int_{\sigma(A)\cap\Omega} |t-\lambda|^2 \, d\mu_\varphi(t) \geq \delta^2 \int_{\sigma(A)\cap\Omega} d\mu_\varphi(t) = \delta^2 \langle \varphi, \varphi \rangle.$$

Thus $\lambda \notin \sigma(A\restriction_{\mathcal{H}_\Omega})$. $\square$

(iv) <u>Multiplication operator form:</u>

Let $A = A^*$ on a separable Hilbert space $\mathcal{H}$. Then there exists a measure space $(M, d\mu)$, with μ a finite measure, and a unitary $U : \mathcal{H} \to L^2(M, d\mu)$ and a function $f : M \longrightarrow \mathbb{R}$ such that

$$U A U^* = T_f .$$

<u>Generalization:</u>

If A_1 and A_2 are two <u>commuting</u> self-adjoint operators, there exist $(M, d\mu)$, $U : \mathcal{H} \to L^2(M, d\mu)$ and two functions $f_1, f_2 : M \to \mathbb{R}$ such that

$$U A_i U^* = T_{f_i} \quad \text{for } i = 1, 2.$$

(If A_1, A_2 do <u>not</u> commute, there is not a joint $(M, d\mu)$ and U that "diagonalizes both of them simultaneously.)

[no proof here]

(v) $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $P_{\{\lambda\}} \neq 0$.
Then $P_{\{\lambda\}}$ is the orthogonal projection on the eigenspace associated to λ.

<u>Proof:</u> let $\varphi \in D(A)$, $\varphi \neq 0$, $(A-\lambda)\varphi = 0$.
Then
$$0 = \|(A-\lambda)\varphi\|^2 = \int |t-\lambda|^2 \, d\mu_\varphi(t).$$
So $\mu_\varphi(\mathbb{R} \setminus \{\lambda\}) = 0$.
Thus
$$0 = \langle \varphi, P_{\mathbb{R}\setminus\{\lambda\}} \varphi \rangle = \langle \varphi, (1 - P_{\{\lambda\}})\varphi \rangle$$
$$= \|(1-P_{\{\lambda\}})\varphi\|^2.$$
Thus $\varphi \in \operatorname{ran} P_{\{\lambda\}}$.

Conversely $\operatorname{ran} P_{\{\lambda\}} \subset D(A)$, thus by Lemma 10.6:
$$(A-\lambda) P_{\{\lambda\}} = \Phi(\underbrace{(I-\lambda)\chi_{\{\lambda\}}}_{=0}) = 0.$$
$\square$

(vi) Instead of $\Phi(f)$ one usually writes $f(A)$, and $P_\Omega(A) = \chi_\Omega(A) = \Phi(\chi_\Omega)$.
If $A\varphi = \lambda\varphi$ then $f(A)\varphi = f(\lambda)\varphi$.

<u>Proof:</u> By (v):
$$f(A)\varphi = f(A) P_{\{\lambda\}}\varphi = \Phi(f\chi_{\{\lambda\}})\varphi$$
$$= \Phi(f(\lambda)\chi_{\{\lambda\}})\varphi = f(\lambda) P_{\{\lambda\}}\varphi$$
$$= f(\lambda)\varphi.$$
$\square$

(vii) In QM, $\mu_\psi(\Omega) = \langle \psi, P_\Omega \psi \rangle$

is the probability to measure a value of the observable A in Ω.

<u>Proposition 10.8:</u> Let $A = A^*$, $P_\Omega(A) := \chi_\Omega(A)$.

Then:

(i) $\lambda \in \sigma(A) \Leftrightarrow P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) \neq 0 \quad \forall \varepsilon > 0$.

(ii) $\lambda \in \sigma_{disc}(A) \Leftrightarrow \lambda \in \sigma(A)$ and

$\dim P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A)\mathcal{H} < \infty$

for some $\varepsilon > 0$.

(iii) $\lambda \in \sigma_{ess}(A) \Leftrightarrow \lambda \in \sigma(A)$ and

$\dim P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A)\mathcal{H} = \infty$

for all $\varepsilon > 0$.

<u>Proof:</u> (i) By the spectral theorem:

$\lambda \in \sigma(A) \Leftrightarrow P_\Omega(A) \neq 0$ for every open $\Omega \ni \lambda$

$\Leftrightarrow P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) \neq 0 \quad \forall \varepsilon > 0$.

(ii) $\lambda \in \sigma_{disc}(A)$ if λ is a isolated eigenvalue of finite multiplicity.

So $P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) = P_{\{\lambda\}}$ for ε small enough.

Thus $\dim P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A)\mathcal{H} < \infty$.

Conversely: If $\lambda \in \sigma(A)$ and

$$n = \dim P_{(\lambda-\varepsilon,\, \lambda+\varepsilon)} \mathcal{H} < \infty.$$

Then by remark (iii) above:

$$\sigma(A) \cap (\lambda-\varepsilon, \lambda+\varepsilon)$$
$$\subset \sigma\left(A\restriction_{P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)\mathcal{H}}\right) = \{\lambda_1, \dots, \lambda_m\}$$
$$m \leq n.$$

So λ is isolated ev of finite multiplicity.

(iii) from (i) & (ii). ∎

Proposition 10.9: Let $A = A^*$, $\sigma(A) = \sigma_1 \cup \sigma_2$, σ_1 compact, $\operatorname{dist}(\sigma_1, \sigma_2) > 0$.
Let γ a C^1-curve circulating once around σ_1 and not intersecting σ_2.

Then

$$P_{\sigma_1} = \frac{1}{2\pi i} \int_\gamma (z-A)^{-1}\, dz.$$

Proof: Exercise. (See complex analysis lecture.) ∎

Examples:

(i) Multiplication operators:

Let $f: \mathbb{R}^n \to \mathbb{R}$ measurable.

T_f in $L^2(\mathbb{R}^n)$ has domain

$$D(T_f) = \{\varphi \in L^2(\mathbb{R}^n) : f\varphi \in L^2(\mathbb{R}^n)\}$$

and acts by

$$(T_f \varphi)(x) = f(x)\,\varphi(x).$$

Show that: for all $g \in \mathcal{B}(\mathbb{R})$:

$$g(T_f) = T_{g \circ f}, \quad \langle \varphi, P_\Omega \varphi \rangle = \int_\Omega |\varphi(x)|^2 dx.$$

Exc.

(ii) Let $A = A^*$, U unitary,

$$B := UAU^{-1} \text{ on } D(B) := UD(A).$$

Show that: Then $B = B^*$ and for all $f \in \mathcal{B}(\mathbb{R})$:

$$f(B) = U f(A) U^{-1}.$$

Exc.