XI Spectral Subspaces and RAGE theorem

Def. 11.1: Let $A = A^*$ on a Hilbert space $\mathcal{H}$. A subspace $M \subset \mathcal{H}$ is called invariant under A if
$$f(A) M \subset M \quad \forall f \in \mathcal{B}(\mathbb{R}).$$

Remark: If M is invariant under A, so is $\overline{M}$.

Lemma 11.2: Let $A = A^*$ on $\mathcal{H}$, $M \subset \mathcal{H}$ a closed subspace and $P \in \mathcal{L}(\mathcal{H})$ the orthogonal projection onto M. Then the following are equivalent:

(a) M is invariant under A.

(b) $M^\perp$ is invariant under A.

(c) $\forall f \in \mathcal{B}(\mathbb{R})$: $f(A)P = P f(A)$.

(d) $PA \subset AP$.

Proof: "(a) $\Rightarrow$ (c)": If $f = f_1 + i f_2$ with $f_k = \overline{f_k}$, then $f(A) = f_1(A) + f_2(A)$, so it is sufficient to show (c) for $\mathbb{R}$-valued $f \in \mathcal{B}(\mathbb{R})$.

By (a) we have $f(A) P \varphi \in M$ $\forall \varphi \in \mathcal{H}$.

Thus $f(A)P = Pf(A)P$. Thus
$$P f(A) = (f(A)P)^* = (P f(A) P)^*$$
$$= P f(A) P = f(A) P.$$

"(c) ⇒ (a)": For all $f \in \mathcal{B}(\mathbb{R})$:

$$f(A)M = f(A)PM$$

$$\overset{(c)}{=} Pf(A)M \subset M,$$

so M is invariant under A.

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From (c) ⇔ (a) it follows that:

$$(c) \Leftrightarrow (1-P)f(A) = f(A)(1-P)$$

$$\Leftrightarrow \operatorname{ran}(1-P) = M^{\perp} \text{ is invariant under } A.$$

Thus (a) ⇔ (b) ⇔ (c).

"(c) ⇒ (d)": Let $e \in D(A)$. Then

$$PAe = i \frac{d}{dt} P e^{-iAt} e \Big|_{t=0}$$

$$\overset{(c)}{=} i \frac{d}{dt} e^{-iAt} Pe \Big|_{t=0}$$

Thus $Pe \in D(A)$ and $APe = PAe$.

"(d) ⇒ (c)": From $PA \subset AP$ we get

$$e^{-iAt} P = P e^{-iAt} \quad \forall t \in \mathbb{R}.$$

(see Proposition on equivalent characterizations of commuting operators).

Thus $f(A)P = Pf(A) \quad \forall f \in \mathcal{B}(\mathbb{R})$

via characterization of the functional calculus in terms of the Fourier transform

$\boxed{/\!/}$

Corollary 11.3: All spectral subspaces $\mathfrak{A}_\Omega = P_\Omega(A)\mathfrak{A}$ are invariant under A.

Definition 11.4:

Let μ a Borel measure on $\mathbb{R}$.

- μ is called <u>continuous</u> if $\mu(\{\lambda\}) = 0$ for all $\lambda \in \mathbb{R}$.
- μ is called <u>pure point measure</u> if there exists a countable $C \subset \mathbb{R}$ with $\mu(\mathbb{R} \setminus C) = 0$.
- μ is called <u>singular with respect to the Lebesgue measure</u> m if there exists a Borel set $N \subset \mathbb{R}$ with $m(N) = 0$ and $\mu(\mathbb{R} \setminus N) = 0$.
- μ is called <u>absolutely continuous with</u>

respect to the Lebesgue measure m if for every Borel set $N \subset \mathbb{R}$

$$m(N) = 0 \Rightarrow \mu(N) = 0.$$

Def. 11.5:

Let $A = A^*$ on $\mathfrak{H}$ and let $\mu_e(\Omega) := \langle e, \chi_\Omega(A) e \rangle$ the spectral measure of $e \in \mathfrak{H}$ w.r.t. A.

We define:

- $\mathfrak{H}_{cont} := \{ e \in \mathfrak{H} : \mu_e \text{ is continuous} \}$
- $\mathfrak{H}_p := \{ e \in \mathfrak{H} : \mu_e \text{ is a pure point measure} \}$
- $\mathfrak{H}_s := \{ e \in \mathfrak{H} : \mu_e \text{ is singular w.r.t. } m \}$
- $\mathfrak{H}_{sc} := \mathfrak{H}_s \cap \mathfrak{H}_{cont}$

  $= \{ e \in \mathfrak{H} : \mu_e \text{ is singular w.r.t. } m \text{ and continuous} \}$
- $\mathfrak{H}_{ac} := \{ e \in \mathfrak{H} : \mu_e \text{ is absolutely continuous w.r.t. } m \}$

Remark: $\mathfrak{H}_p \subset \mathfrak{H}_s$, $\mathfrak{H}_{ac} \subset \mathfrak{H}_{cont}$, $\mathfrak{H}_p$ contains all eigenvectors of A.

Remark: For every Borel set $\Omega \subset \mathbb{R}$ and all $e \in \mathfrak{H}$:

$$\mu_e(\mathbb{R} \setminus \Omega) = 0 \Leftrightarrow P_\Omega e = e.$$

In fact, $\mu_e(\mathbb{R} \setminus \Omega)$

$$= \langle e, \chi_{\mathbb{R}\setminus\Omega}(A) e\rangle$$
$$= \langle e, (1-\chi_\Omega)(A) e\rangle$$
$$= \|(1-P_\Omega) e\|^2.$$

This implies:

$\mathfrak{H}_p = \{e \in \mathfrak{H} : \exists C \subset \mathbb{R}$ countable with $P_C e = e\}$

$\mathfrak{H}_s = \{e \in \mathfrak{H} : \exists N \subset \mathbb{R}$ Borel with $m(N) = 0$ and $P_C e = e\}$.

Proposition 11.6: $\mathfrak{H}_p = \overline{\mathrm{span}\{e \in \mathfrak{H} \text{ is eigenvector of } A\}}$

In particular $\mathfrak{H}_p$ is invariant under A.

Proof: (i) We show that $\mathfrak{H}_p$ is a subspace.

Let $e_1, e_2 \in \mathfrak{H}_p$. Then there exist countable sets $C_1, C_2 \subset \mathbb{R}$ with $P_{C_1} e_1 = e_1$ and $P_{C_2} e_2 = e_2$.

Also $C_1 \cup C_2$ is countable and

$P_{C_1 \cup C_2}(e_1 + e_2) = e_1 + e_2$

$P_{C_1}(\alpha e_1) = \alpha P_{C_1} e_1 = \alpha e_1, \ \alpha \in \mathbb{C}$.

(ii) We show that $\mathfrak{H}_p$ is closed.

Let $(e_n)_n$ a sequence in $\mathfrak{H}_p$ with $e_n \to e$.

Then there exist countable sets C_n with $P_{C_n} e_n = e_n$.

Also $C := \bigcup_{n \in \mathbb{N}} C_n$ is countable, and

$$P_C \varphi = \lim_{n\to\infty} P_C \varphi_n = \lim_{n\to\infty} \varphi_n = \varphi .$$

Thus $\mathcal{H}_p$ is closed.

(iii) Clearly $\mathcal{H}_p$ contains all eigenvectors of A.

Thus
$$\mathcal{H}_p \supset \overline{\operatorname{span}\{\varphi \in D(A) : A\varphi = \lambda\varphi, \lambda \in \mathbb{R}\}}$$

(iv) Let $\varphi \in \mathcal{H}_p$. By the above remark there exists a countable $C \subset \mathbb{R}$ with $P_C \varphi = \varphi$.

Write $C = \{\lambda_n : n \in \mathbb{N}\}$.

Then $P_C = \text{s-}\lim_{N\to\infty} \sum_{i=1}^{N} P_{\{\lambda_i\}}$ (see Chapter 10).

Thus
$$\varphi = P_C \varphi = \lim_{N\to\infty} \sum_{i=1}^{N} P_{\{\lambda_i\}} \varphi ,$$

where $A P_{\{\lambda_i\}} \varphi = \lambda_i P_{\{\lambda_i\}} \varphi$.

Therefore $\varphi \in \overline{\operatorname{span}\{\varphi \in D(A) : A\varphi = \lambda\varphi, \lambda \in \mathbb{R}\}}$

Moreover for all $f \in \mathcal{B}(\mathbb{R})$:

$$f(A)\varphi = f(A) P_C \varphi = P_C f(A) \varphi .$$

Thus $f(A)\varphi \in \mathcal{H}_p$. ■

Recall: If $X_1, \dots X_N$ are subspaces of a vector space X, one writes

$$X = X_1 \oplus X_2 \oplus \dots \oplus X_N$$

if every vector $e \in \mathcal{H}$ has a unique decomposition

$$e = e_1 + \dots + e_N, \quad e_i \in X_i.$$

If X is a Banach space and $X_1, \dots X_N$ are closed subspaces, then X is isomorphic to the cartesian product (N-tuples)

$$\prod_{i=1}^{N} X_i, \quad \|(e_1, \dots e_N)\| = \sum_{i=1}^{N} \|e_i\|.$$

In particular, the norms $\|e\|$ and $\sum_{i=1}^{N} \|e_i\|$ are equivalent.

(i.e., there exist c_N and d_N such that

$$c_N \|e\| \le \sum_{i=1}^{N} \|e_i\| \le d_N \|e\| \quad \forall e.)$$

Theorem 11.7: $\mathcal{H}_{cont}, \mathcal{H}_p, \mathcal{H}_s, \mathcal{H}_{sc}, \mathcal{H}_{ac}$ are closed subspaces of $\mathcal{H}$, invariant under A, and

$$\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_{cont} = \mathcal{H}_s \oplus \mathcal{H}_{ac}$$
$$= \mathcal{H}_p \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{ac}.$$

$\mathcal{H}_p$, $\mathcal{H}_{sc}$, $\mathcal{H}_{ac}$ are pairwise orthogonal.

Proof: (i) We show $\mathcal{H}_{cont} = \mathcal{H}_p^{\perp}$.

Let $e \in \mathcal{H}_p$, $\gamma \in \mathcal{H}_{cont}$.

Then there exists a countable set $C \subset \mathbb{R}$ with $P_C e = e$. Thus

$$\langle e, \gamma \rangle = \langle P_C e, \gamma \rangle = \langle e, P_C \gamma \rangle = 0$$

because

$$\| P_C \gamma \|^2 = \langle \gamma, P_C \gamma \rangle = \sum_{\lambda \in C} \mu_\gamma(\{\lambda\}) = 0$$

because $\gamma \in \mathcal{H}_{cont}$.

Now let $\gamma \perp \mathcal{H}_p$ and $\lambda \in \mathbb{R}$.

Then

$$\mu_\gamma(\{\lambda\}) = \langle \gamma, P_{\{\lambda\}} \gamma \rangle = 0$$

because $P_{\{\lambda\}} \gamma \in \mathcal{H}_p$.

So $\gamma \in \mathcal{H}_{cont}$.

By Lem. 11.2 and Proposition 11.6, $\mathcal{H}_{cont}$ is a closed subspace invariant under A.

(ii) We show that $\mathfrak{H}_s$ is a closed subspace, invariant under A, using the same argument as in proof of Prop. 11.6. (152)

(iii) We show $\mathfrak{H}_{ac} = \mathfrak{H}_s^\perp$.

Let $\varphi \in \mathfrak{H}_s$, $\gamma \in \mathfrak{H}_{ac}$.
There exists a Lebesgue null set N with $P_N \varphi = \varphi$. Therefore
$$\langle \varphi, \gamma \rangle = \langle P_N \varphi, \gamma \rangle = \langle \varphi, P_N \gamma \rangle = 0$$
because
$$\|P_N \gamma\|^2 = \langle \gamma, P_N \gamma \rangle = \mu_\gamma(N) = 0$$
because $\gamma \in \mathfrak{H}_{ac}$.

Now let $\gamma \in \mathfrak{H}_s^\perp$, $N \subset \mathbb{R}$ a Lebesgue null set. Then $P_N \gamma \in \mathfrak{H}_s$.
Thus $\mu_\gamma(N) = \langle \gamma, P_N \gamma \rangle = 0$
so $\gamma \in \mathfrak{H}_{ac}$.

(i) $\wedge$ (ii) $\wedge$ (iii) $\Rightarrow$ $\mathfrak{H}_p, \mathfrak{H}_{sc}, \mathfrak{H}_{ac}$ are pairwise orthogonal.

(IV) We show $\mathcal{H}_s = \mathcal{H}_p \oplus \mathcal{H}_{sc}$. (153)

Let $e \in \mathcal{H}_s$, and $e = e_1 + e_2$ with $e_1 \in \mathcal{H}_p$ and $e_2 \in \mathcal{H}_{cont}$.

Then

$$e_2 = e - e_1 \in \mathcal{H}_s \cap \mathcal{H}_{cont} = \mathcal{H}_{sc}$$

because $e_1 \in \mathcal{H}_p \subset \mathcal{H}_s$.

For uniqueness, let $e = e_1' + e_2'$ a second decomposition with $e_1' \in \mathcal{H}_p$ and $e_2' \in \mathcal{H}_{sc}$. Then

$$e_1' + e_2' = e_1 + e_2$$

$$\Rightarrow e_1' - e_1 = e_2 - e_2' \in \mathcal{H}_p \cap \mathcal{H}_{sc} = \{0\}.$$

Thus $e_1' = e_1$ and $e_2' = e_2$.

$\blacksquare$

Remark: Compare this chapter to the Radon-Nikodym theorem and Lebesgue's decomposition theorem.