Def. 11.8: We define the sets

$\sigma_p(A) := \{\text{eigenvalues of } A\}$ point spectrum

$\sigma_{cont}(A) := \sigma(A\restriction_{\mathcal{H}_{cont}})$ continuous spectrum

$\sigma_s(A) := \sigma(A\restriction_{\mathcal{H}_s})$ singular spectrum

$\sigma_{sc}(A) := \sigma(A\restriction_{\mathcal{H}_{sc}})$ singular continuous spectrum

$\sigma_{ac}(A) := \sigma(A\restriction_{\mathcal{H}_{ac}})$ absolutely continuous spectrum.

Remark: $\overline{\sigma_p(A)} = \sigma(A\restriction_{\mathcal{H}_p})$

and therefore

$$\sigma(A) = \overline{\sigma_p(A)} \cup \sigma_{cont}(A)$$
$$= \sigma_s(A) \cup \sigma_{ac}(A)$$
$$= \overline{\sigma_p(A)} \cup \sigma_{sc}(A) \cup \sigma_{ac}(A)$$

⚠ but these are in general <u>not</u> disjoint.

The decomposition of the spectrum is invariant under unitary transformations:

Proposition 11.8: Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ unitary, $A = A^*$ in $\mathcal{H}_1$, $B = UAU^{-1}$, $D(B) := U\,D(A)$. Then, for $\# \in \{p, cont, s, sc, ac\}$:

$$\mathcal{H}_\#(B) = U\,\mathcal{H}_\#(A),\quad \sigma_\#(B) = \sigma_\#(A).$$

Proof: Exercise. ∎

Example: Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $H = -\Delta$ on $D(H) = H^2(\mathbb{R}^n)$. Then $\mathcal{H}_{ac} = L^2(\mathbb{R}^n)$.

Proof: By Prp. 11.8 it is sufficient to show $\mathcal{H} = \mathcal{H}_{ac}(T_f)$, $f(p) = p^2$

Let $\varphi \in L^2(\mathbb{R}^n)$ and $\Omega \subset [0, \infty)$ a Lebesgue null set. Then

$$\langle \varphi, \chi_\Omega(T_f)\varphi\rangle = \langle \varphi, T_{\chi_\Omega \circ f}\,\varphi\rangle$$

by uniqueness of the functional calculus.

$$= \int_{\mathbb{R}^n} \chi_\Omega(f(x))\, |\varphi(x)|^2 \, dx$$

$$= \int_{f^{-1}(\Omega)} |\varphi(x)|^2 \, dx ,$$

which is $= 0$ because we now show that $m(f^{-1}(\Omega)) = 0$.

In fact, let $\tilde{\Omega} := \{ r \geq 0 \,|\, r^2 \in \Omega \}$.

Then $m(\tilde{\Omega}) = \int_\Omega \frac{1}{2\sqrt{r}} \, dr = 0$.

Thus

$$m(f^{-1}(\Omega))$$

$$= m(\{ x \in \mathbb{R}^n : x^2 \in \Omega \})$$

$$= m(\{ x \in \mathbb{R}^n : |x| \in \tilde{\Omega} \})$$

$$= \int_{\sqrt{\Omega}} r^{n-1} \underbrace{|S^{n-1}|}_{\substack{\text{surface area of the} \\ \text{unit sphere } S^{n-1} \text{ in } \mathbb{R}^n.}} \, dr = 0$$

$\blacksquare$

Proposition 11.10: (Wiener)

Let μ be a finite Borel measure (i.e. $\mu(\mathbb{R}) < \infty$)

on $\mathbb{R}$, $C := \{x \in \mathbb{R} \mid \mu(\{x\}) > 0\}$

and $F(t) := \int_{\mathbb{R}} e^{-ixt} \, d\mu(x)$.

Then C is countable and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |F(t)|^2 \, dt = \sum_{x \in C} |\mu(x)|^2 .$$

Proof: Since $\mu(\mathbb{R}) < \infty$

also for

$$C_n := \{x \in \mathbb{R} \mid \mu(\{x\}) > \tfrac{1}{n}\}$$

we have $\mu(C_n) < \infty$,

therefore C_n is a finite set.

Thus $C = \bigcup_{n \geq 1} C_n$ is countable.

We have

$$|F(t)|^2 = \int e^{iyt}\, d\mu(y) \int e^{-ixt}\, d\mu(x)$$
$$= \int d\mu(y) \int d\mu(x)\, e^{i(y-x)t}.$$

By Fubini–Tonelli we can exchange integrals:

$$\frac{1}{T}\int_0^T |F(t)|^2\, dt = \int d\mu(y) \int d\mu(x)\, \frac{1}{T}\int_0^T e^{i(y-x)t}\, dt$$

For $x \neq y$:

$$g(T) := \frac{1}{T}\int_0^T e^{i(y-x)t}\, dt \overset{\substack{\text{explicit}\\ \text{integration}}}{=} \frac{e^{i(y-x)T} - 1}{i(y-x)T},$$

$$|g(T)| \leq \frac{|e^{i(y-x)T}| + 1}{|y-x|\, T} = \frac{2}{|y-x|} \cdot \frac{1}{T} \longrightarrow 0$$

as $T \to \infty$.

For $x = y$:

$$g(T) = \frac{1}{T}\int_0^T 1\, dt = 1.$$

So $\lim_{T\to\infty} g(T) = \chi_{\{y\}}(x)$.

Using Lebesgue's dominated convergence theorem twice we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |F(t)|^2\, dt$$

$$= \int d\mu(y) \int d\mu(x)\, \chi_{\{y\}}(x) = \int d\mu(y)\, \mu(\{y\})$$

$$= \int_C d\mu(y)\, \mu(\{y\}) = \sum_{y\in C} \mu(\{y\})^2.$$

$\blacksquare$

Theorem 11.11: (Ruelle-Amrein-Georgescu-Enss)

Let $\mathcal{H}$ a separable Hilbert space, $A = A^*$ on $\mathcal{H}$, $C \in \mathcal{L}(\mathcal{H})$ and $C(A+i)^{-1}$ compact. Then for all $\varphi \in \mathcal{H}_{cont}(A)$:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \| C e^{-iAt}\varphi\|^2\, dt = 0.$$

Proof:

1) If C is rank-one, $C\varphi = \gamma\langle\psi,\varphi\rangle$ for some $\gamma, \psi \in \mathcal{H}$: Then

$$\|C e^{-iAt}\varphi\| = \|\gamma\langle\psi, e^{-iAt}\varphi\rangle\|$$

$$= \|\gamma\|\, |\langle\psi, e^{-iAt}\varphi\rangle|.$$

Since $\varphi \in \mathfrak{H}_{cont} = P_{cont}\mathfrak{H}$: (160)

$$\langle \psi, e^{-iAt}\varphi\rangle = \langle \psi, e^{-iAt}P_{cont}\varphi\rangle$$

Since P_{cont} is a spectral projection of A, it commutes with e^{-iAt}.

$$= \langle P_{cont}\psi, e^{-iAt}\varphi\rangle$$

Thus we can assume that w.l.o.g. $\psi \in \mathfrak{H}_{cont}$.

By the polarization identity:

$$\langle \psi, e^{-iAt}\varphi\rangle = \frac{1}{4}\langle \psi+\varphi, e^{-iAt}(\psi+\varphi)\rangle - \frac{1}{4}\langle \psi-\varphi, e^{-iAt}(\psi-\varphi)\rangle + \frac{1}{4i}\langle \psi+i\varphi, e^{-iAt}(\psi+i\varphi)\rangle - \frac{1}{4i}\langle \psi-i\varphi, e^{-iAt}(\psi-i\varphi)\rangle.$$

By Cauchy-Schwartz, for all $\alpha_i \in \mathbb{C}$:

$$\left|\sum_{i=1}^{4}\alpha_i\right|^2 = \left|\sum_{i=1}^{4} 1\cdot\alpha_i\right|^2 \le \left|\left(\sum_{i=1}^{4}1^2\right)^{1/2}\left(\sum_{i=1}^{4}|\alpha_i|^2\right)^{1/2}\right|^2 = 4\cdot\sum_{i=1}^{4}|\alpha_i|^2.$$

It is therefore sufficient to show the claim for $\varphi = e \in \mathfrak{H}_{\text{cont}}$.

Let $F(t) := \langle e, e^{-iAt} e \rangle$

$$= \int e^{-ixt} \, d\mu_e(x).$$

By Wiener's proposition (Proposition 11.10) then

$$\frac{1}{T} \int_0^T |\langle e, e^{-iAt} e \rangle|^2 \, dt \longrightarrow \sum_{x \in C'} |\mu_e(\{x\})|^2,$$

but since $e \in \mathfrak{H}_{\text{cont}}$, μ_e is a continuous measure, therefore $C' = \emptyset$.

So the limit is zero, as had to be shown to complete the first step.

Step 2: C finite rank.

Step 3: From finite rank to compact.

Step 4: From compact to relatively compact.

ne lect

[...]