2) If C is finite-rank, $C\varphi = \sum_{n=1}^{N} \psi_n \langle e_n, \varphi\rangle$

Then

$$\|C e^{-iAt}\varphi\|^2 \leq \left(\sum_{n=1}^{N} \|\psi_n \langle e_n, e^{-iAt}\varphi\rangle\| \right)^2$$

and by Cauchy-Schwartz $\leq N \sum_{n=1}^{N} \|\psi_n \langle e_n, e^{-iAt}\varphi\rangle$

[illegible] is just Case 1.

Now we take the closure of finite-rank operators, which is compact operators $\mathcal{C}$.

3) $C \in \mathcal{L}(\mathcal{H})$ compact.

Let $(e_n)_{n \in \mathbb{N}}$ a ONB of $\mathcal{H}$ and

$$P_N \varphi := \sum_{n=1}^{N} e_n \langle e_n, \varphi \rangle .$$

Then $\| C P_N - C \| \longrightarrow 0 \quad (n \to \infty)$

and $$C P_N \varphi = \sum_{n=1}^{N} C e_n \langle e_n, \varphi \rangle .$$

Thus, by Case 2:

$$\frac{1}{T} \int_0^T \| C P_N e^{-iAt} \varphi \|^2 dt \to 0 \quad (T \to \infty)$$

Now let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large enough that $\| C P_N - C \|^2 \, \|\varphi\|^2 < \frac{\varepsilon}{4}$.

Then choose $T_0 \in \mathbb{R}$ large enough that $\frac{1}{T} \int_0^T \| C P_N e^{-iAt} \varphi \|^2 dt < \frac{\varepsilon}{4}$ for all $T > T_0$.

Then

$$\frac{1}{T}\int_0^T \|C e^{-iAt} e\|^2 \, dt$$

$$= \frac{1}{T}\int_0^T \|(C - CP_N + CP_N) e^{-iAt} e\|^2 dt$$

using $(A+B)^2 \le 2A^2 + 2B^2$ for all $A, B \in \mathbb{R}$

$$= \frac{2}{T}\int_0^T \|(C - CP_N) e^{-iAt} e\|^2 \, dt$$

$$+ \underbrace{\frac{2}{T}\int_0^T \|CP_N e^{-iAt} e\|^2 \, dt}_{2\cdot\frac{\varepsilon}{4}}$$

by unitarity

$$\le 2 \underbrace{\|C - CP_N\|^2 \|e\|^2}_{< \frac{\varepsilon}{4}} \underbrace{\frac{1}{T}\int_0^T dt}_{=1} + 2\frac{\varepsilon}{4}$$

$$< \varepsilon .$$

4) <u>$C \in \mathcal{L}(\mathcal{H})$, $C(A+i)^{-1}$ compact.</u>

For $e \in D(A) \cap \mathcal{H}_{cont}$, by the previous step:

$$\frac{1}{T}\int_0^T \|C e^{-iAt} e\|^2 \, dt$$

$$= \frac{1}{T}\int_0^T \|C(A+i)^{-1} e^{-iAt} \underbrace{(A+i) e}_{\in \mathcal{H}_{cont}}\|^2 \, dt \longrightarrow 0 \quad (T \to \infty)$$

Since $D(A) \subset \mathfrak{H}$ and $\|C\| < +\infty$ the extension from $\varphi \in D(A) \cap \mathfrak{H}_{cont}$ to $\varphi \in \mathfrak{H}_{cont}$ follows by an approximation argument just as in the previous step. ∎

We now think of A as a Hamiltonian and take $C := \chi_R(x)$, the multiplication operator given by characteristic function of the ball $\{x \in \mathbb{R}^n : |x| \leq R\}$

<u>Theorem 11.12:</u> (RAGE on L^2)

Let $H = H^*$ in $\mathfrak{H} = L^2(\mathbb{R}^n)$ and let $\chi_R (H+i)^{-1}$ compact for all $R > 0$. Then

$$\varphi \in \mathfrak{H}_p \Leftrightarrow \lim_{R \to \infty} \|(1-\chi_R) e^{-iHt} \varphi\| = 0 \quad \text{uniformly in } t \geq 0$$

$$\varphi \in \mathfrak{H}_{cont} \Leftrightarrow \lim_{T \to \infty} \frac{1}{T} \int_0^T \|\chi_R e^{-iHt} \varphi\|^2 dt = 0 \quad \text{for all } R > 0.$$

Proof: Define

$$\mathcal{H}_1 := \{\varphi \in \mathcal{H} \mid \lim_{R\to\infty} \|(1-\chi_R)e^{-iHt}\varphi\| = 0 \text{ uniformly in } t \geq 0\}$$

$$\mathcal{H}_2 := \{\varphi \in \mathcal{H} \mid \lim_{T\to\infty} \frac{1}{T}\int_0^T \|\chi_R e^{-iHt}\varphi\|^2\, dt = 0 \text{ for all } R > 0\}.$$

It is trivial to see that $\mathcal{H}_p \subset \mathcal{H}_1$,

and by the RAGE theorem $\mathcal{H}_{cont} \subset \mathcal{H}_2$.

Since $\mathcal{H}_{cont} = \mathcal{H}_p^{\perp}$, the opposite inclusions follow if $\mathcal{H}_1 \perp \mathcal{H}_2$; in fact

$$\varphi \in \mathcal{H}_1 \Rightarrow \varphi \perp \mathcal{H}_2 \Rightarrow \varphi \perp \mathcal{H}_{cont} \Rightarrow \varphi \in \mathcal{H}_p$$

$$\varphi \in \mathcal{H}_2 \Rightarrow \varphi \perp \mathcal{H}_1 \Rightarrow \varphi \perp \mathcal{H}_p \Rightarrow \varphi \in \mathcal{H}_{cont}$$

To show $\mathcal{H}_1 \perp \mathcal{H}_2$, let $\varphi_1 \in \mathcal{H}_1$ and $\varphi_2 \in \mathcal{H}_2$.

Then for all $T, R > 0$:

by unitarity

$$\langle \varphi_1, \varphi_2 \rangle = \overbrace{\frac{1}{T}\int_0^T}^{=1} \langle \cancel{e^{-iHt}}\varphi_1, \cancel{e^{-iHt}}\varphi_2 \rangle\, dt$$

$$= \frac{1}{T}\int_0^T \langle (1-\chi_R)e^{-iHt}\varphi_1, e^{-iHt}\varphi_2 \rangle\, dt$$

$$+ \frac{1}{T}\int_0^T \langle e^{-iHt}\varphi_1, \chi_R e^{-iHt}\varphi_2 \rangle\, dt.$$

Therefore

$$|\langle e_1, e_2\rangle| \leq \frac{1}{T}\int_0^T |\langle (1-\chi_R)e^{-iHt}e_1, e^{-iHt}e_2\rangle dt$$

$$+ \frac{1}{T}\int_0^T |\langle e^{-iHt}e_1, \chi_R e^{-iHt}e_2\rangle dt|$$

$$\leq \sup_{t\geq 0} \|(1-\chi_R)e^{-iHt}e_1\| \, \|e_2\|$$

$$+ \frac{1}{T}\int_0^T \|\chi_R e^{-iHt}e_2\| \, dt \; \|e_1\|$$

Let $\varepsilon > 0$. Pick $R \in \mathbb{R}$ large enough that $\|(1-\chi_R)e^{-iHt}e_1\| \, \|e_2\| < \frac{\varepsilon}{2}$. By Cauchy-Schwartz: $\frac{1}{T}\int_0^T 1 \cdot \|\chi_R e^{-iHt}e_2\| dt$

$$\leq \underbrace{\left(\frac{1}{T}\int_0^T 1 \, dt\right)^{1/2}}_{=1} \left(\frac{1}{T}\int_0^T \|\chi_R e^{-iHt}e_2\|^2 dt\right)^{1/2},$$

pick $T \in \mathbb{R}$ large enough that

$$\frac{1}{T}\int_0^T \|\chi_R e^{-iHt}e_2\| \, dt \; \|e_1\| < \frac{\varepsilon}{2}.$$

Then, for R, T large enough

$$|\langle e_1, e_2\rangle| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for all $\varepsilon > 0$, so

$$\langle e_1, e_2 \rangle = 0. \qquad \blacksquare$$

Application: $H = -\Delta + V$, if V

satisfies the assumptions of the Kato-Rellich theorem. In fact,

$$\chi_R (H+i)^{-1} = \underbrace{\chi_R(-\Delta+i)^{-1}}_{\substack{\text{compact} \\ \text{(Kato-Seiler-Simon)}}} \underbrace{(-\Delta+i)(H+i)^{-1}}_{\substack{\text{closed} \\ \text{and everywhere} \\ \text{defined}}}$$

because

$(H+i)^{-1} : L^2(\mathbb{R}^n) \to D(-\Delta)$

thus bounded.

Interpretation: Vectors in $\mathfrak{H}_p$ are bound states: With probability $(1-\varepsilon)$ they stay for all times $t \geq 0$ in a ball $B_{R_\varepsilon}(0)$.

Vectors in $\mathfrak{H}_{cont}$ are unbounded: with probability 1, in time-average, they leave every ball B_R.

Possible further topics:

- perturbation theory: see, e.g., Reed & Simon
- asymptotic completeness of Schrödinger operators with short-range potentials using RAGE theorem, see, e.g., Teschl § 12.3
- second quantization, many-body systems, quantum field theory, see, e.g., PhD course
- entanglement, quantum information theory
- quantum statistical mechanics
- Anderson localization
- ...