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Metodi Matematici della Meccanica Quantistica

Assignment 1

To be handed in on **Friday, October 11, 2024 before 10:30** via email (scan, readable foto, or LaTeX) to ngoc.nguyen@unimi.it.

Problem 1: Banach Spaces and Bounded Operators (3+4+3 points)

a. Let X be a Banach space and $\tilde{X} \subset X$ a closed subspace. Show that $\tilde{X}$ is a Banach space.

b. Let X be a normed vector space and Y be a Banach space. Show that the space of bounded linear operators from X to Y, i. e., $\mathcal{L}(X, Y)$ with the operator norm, is a Banach space.

c. Let X, Y be Banach spaces and $A : D \to Y$ a bounded linear operator defined on a dense subspace $D \subset X$. Construct an extension $\overline{A}$ of A to all of X such that the extension is a linear operator with the same operator norm as A, i. e., $\|\overline{A}\| = \|A\|$.

Problem 2: Derivative Operator (5+5 points)

Consider (as in the example presented in the lecture) the Banach space $X = C([0, 1])$ of continuous functions with the norm $\|f\| := \sup_{x\in[0,1]}|f(x)|$.

Define operators $A_k : D_k \subset X \to X$ (for the two options $k \in \{3, 4\}$) by the mapping

$$(A_k f)(x) := f'(x)$$

with domains given as subsets of the continuously differentiable functions as follows

$$\begin{aligned} D_3 &:= \{f \in C^1([0, 1]) \mid f(0) = f(1)\} \qquad &\text{(periodic boundary conditions)}\ , \\ D_4 &:= \{f \in C^1([0, 1]) \mid f(0) = 0 = f(1)\} \qquad &\text{(Dirichlet boundary conditions)}\ . \end{aligned}$$

Show that

a. $\sigma(A_3) = \sigma_{\mathrm{p}}(A_3) = 2\pi i\mathbb{Z}$,

b. $\sigma(A_4) = \mathbb{C}$ and $\sigma_{\mathrm{p}}(A_4) = \emptyset$.

Problem 3: Liouville's theorem (10 points)

Prove the following theorem:

Let X be a Banach space. Let $L : \mathbb{C} \to \mathcal{L}(X)$ be an operator-valued entire function. If the operator norm of L is bounded, i. e., if there exists $M \in \mathbb{R}$ such that $\|L(z)\| \leq M$ for all $z \in \mathbb{C}$, then L is constant.