Prof. Dr. Niels Benedikter
Dr. Ngoc Nhi Nguyen

Metodi Matematici della Meccanica Quantistica

Assignment 2

To be handed in on **Wednesday, October 23, 2024 before 20:59** via email (scan, readable foto, or LaTeX) to ngoc.nguyen@unimi.it.

Problem 1: Operator Adjoints (5+5 points)

a. Let $\mathcal{H}$ be a Hilbert space and A, B densely defined operators in $\mathcal{H}$.
Show that: If $A \subset B$, then $B^* \subset A^*$.

b. Let $\mathcal{H}$ be a Hilbert space and A, B densely defined operators in $\mathcal{H}$.
Show that: If A is symmetric and B is a self-adjoint extension of A, then $A \subset B \subset A^*$.

Problem 2: Polarization Identity (6 points)

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V and $\|x\| := \sqrt{\langle x, x \rangle}$ the norm it induces. Show that for all $x, y \in V$ we have

$$\langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2 \right) .$$

Remark: This permits to recover the scalar product from the norm.

Problem 3: Operator with trivial adjoint (3+4+3 points)

A countable family $(e_n)_{n \in \mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called **orthonormal basis** or **Schauder basis** if $\langle e_n, e_k \rangle = \delta_{n,k}$ and for all $x \in \mathcal{H}$ we have *as a convergent infinite series*

$$x = \sum_{n=0}^{\infty} e_n \langle e_n, x \rangle . \tag{1}$$

Remark: This is different from linear algebra where a basis uses only finite linear combinations. A basis that can represent any vector in terms of finite linear combinations is called Hamel basis but is not commonly used in Hilbert spaces. One reason is that, if X is an infinite-dimensional Banach space, then any Hamel basis of X is uncountable.

Example: For $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$, in $\mathcal{H} = L^2((0, 2\pi))$, the series (1) is the ordinary Fourier series.

Now let $\mathcal{H} := L^2(\mathbb{R})$ and $(e_n)_{n\in\mathbb{N}}$ an arbitrary orthonormal basis of $\mathcal{H}$. Define an operator $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ by $\mathcal{D} := C_0^\infty(\mathbb{R})$ and

$$Af := \sum_{n=0}^{\infty} f(n) e_n \,. \tag{2}$$

a. Show that: The series in (2) converges.

b. Show that: For any $g \in \mathcal{H}$, $g \neq 0$, the mapping $f \mapsto \langle g, Af\rangle$ is not continuous as a function from $(\mathcal{D}, \|\cdot\|_{\mathcal{H}})$ to $\mathbb{C}$.

c. Show that the domain $D(A^*)$ is trivial, i. e., $D(A^*) = \{0\}$.

Problem 4: Orthogonal Complement (4+4 points)

Let $\mathcal{H}$ be a Hilbert space and $M \subset \mathcal{H}$ a subset. Prove the following facts:

a. The orthogonal complement $M^\perp$ is a closed subspace.

b. $M \subset (M^\perp)^\perp$. If M is a subspace: $(M^\perp)^\perp = \overline{M}$.

You may use that given a closed subspace $X \subset \mathcal{H}$, for every $x \in \mathcal{H}$ there exists a unique decomposition $x = x_1 + x_2$ with $x_1 \in X$ and $x_2 \in X^\perp$.