Prof. Dr. Niels Benedikter
Dr. Ngoc Nhi Nguyen

Metodi Matematici della Meccanica Quantistica

Assignment 4

To be handed in on **Wednesday, December 18, 2024, before 23:59** via email (scanned or LaTeX) to ngoc.nguyen@unimi.it.

Problem 1: Method of Stationary Phase (10 points)

Let $\omega \in C^\infty(\mathbb{R}^n)$ and $u \in C_0^\infty(\mathbb{R}^n)$. Let G be an open neighbourhood of $\{\nabla\omega(k) \mid k \in \operatorname{supp}(u)\}$. For simplicity consider only $n = 1$ in the following (but note that the result holds for any $n \in \mathbb{N}$).

Show that for all $m \in \mathbb{N}$ there exists $c_m \in \mathbb{R}$ such that

$$\left| \int_{\mathbb{R}^n} e^{i(k\cdot x - \omega(k)t)} u(k) \mathrm{d}k \right| \leq \frac{c_m}{(1+|t|)^m} \quad \text{for all } x \text{ and } t \text{ satisfying } \frac{x}{t} \notin G.$$

Heuristics: This quantifies the intuition that positive and negative parts of the integral cancel if the phase is rapidly oscillating. The integral “survives” only when the phase is stationary (when $x/t \in G$).

Hint: Let $\phi(k) := k\frac{x}{t} - \omega(k)$. Use $e^{i\phi(k)t} = \left(\frac{1}{i\phi'(k)t}\frac{\mathrm{d}}{\mathrm{d}k}\right)^m e^{i\phi(k)t}$ and integration by parts.

Problem 2: Compact Operators (10 points)

Let X, Y, Z be Banach spaces. Let $K \in \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(Y, Z)$. Let moreover K or L be compact. Show that in both cases LK is compact.

Problem 3: Kato–Seiler–Simon for $p = 2$ (10 points)

Let $f, g \in L^2(\mathbb{R}^n)$ and let $B := f(x)g(-i\nabla)$. Show that $\|B\| \leq (2\pi)^{-n/2}\|f\|_{L^2}\|g\|_{L^2}$ and $\|B\| \leq \|f\|_{L^\infty}\|g\|_{L^\infty}$.

Problem 4: Essential Spectrum of $-\Delta$ (10 points)

Consider $-\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Show that $\sigma_{\text{ess}}(-\Delta) \supset [0, \infty)$.