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Metodi Matematici della Meccanica Quantistica

Last Assignment

To be handed in on **Wednesday, January 8, 2025, before 23:59** via email (scanned or LaTeX) to ngoc.nguyen@unimi.it.

Problem 1: Wave Operators for Pseudo-Relativistic Particle (15 points)

Let $\omega(p) := \sqrt{p^2+1}$ and consider the pseudo-relativistic Hamiltonian

$$H = H_0 + V \ , \quad H_0 = \sqrt{-\Delta + 1} := \mathcal{F}^{-1} T_\omega \mathcal{F}$$

where the potential $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies, for some $\mu > 1$, the decay $|V(x)| \leq \text{const} \cdot |x|^{-\mu}$ for $|x| > R$. You can take for granted that $H = H^*$ on $D(H_0)$.

Show that the wave operators $\Omega_\pm$ exist.

Hint: Let $D = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid \hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\}$. Let $\varphi \in D$ with $\text{supp}(\hat{\varphi}) \subset \{|p| \geq \varepsilon\}$. Then for all $p \in \text{supp}(\hat{\varphi})$ we have by monotonicity

$$|\nabla \omega(p)| = \frac{|p|}{\sqrt{p^2+1}} \geq \frac{\varepsilon}{\sqrt{\varepsilon^2+1}} =: 2\delta \ .$$

Decompose the potential into parts $|x| \leq \delta t$ and $|x| > \delta t$.

Problem 2: Abelian Limits (5 + 5 + 5 points)

a. Let $H_0 = -\Delta/2$ in $L^2(\mathbb{R}^3)$ and $E \in \mathbb{R}$. Show that

$$\operatorname*{s-lim}_{\varepsilon \downarrow 0} \varepsilon (H_0 - E + i\varepsilon)^{-1} = 0.$$

Hint: Use the Fourier transform and choose a convenient dense subspace.

b. Let $\varphi : [0, \infty) \to X$ be continuous, X a Banach space and assume that $\varphi_\infty := \lim_{t\to\infty} \varphi(t)$ exists. Prove that

$$\varphi_\infty = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} \varphi(t) \mathrm{d}t \ .$$

c. Let H be a self-adjoint operator in $L^2(\mathbb{R}^3)$ with $D(H) = D(H_0)$, $H_0 = -\Delta/2$, and assume that the wave operator $\Omega_+ = \text{s-lim}_{t\to\infty} e^{iHt} e^{-iH_0 t}$ exists. Assume furthermore that asymptotic completeness holds, i. e., $\operatorname{ran}\Omega_+ = \mathcal{H}_B^\perp$, where $\mathcal{H}_B$ is the closure of the span of the eigenstates of H.

Prove that for all $\varphi \in \mathcal{H}$ we have

$$\Omega_+^* \varphi = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} e^{iH_0 t} e^{-iHt} \varphi \, \mathrm{d}t \; .$$

Problem 3: Concatenation and Functional Calculus (15 points)

Let A be a self-adjoint operator on a Hilbert space. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. Is it true that $f(g(A)) = (f \circ g)(A)$? Provide a proof or a counterexample.

Problem 4: An Integral Resolvent Representation (5+5+5+5 points)

Let $A = A^*$ be an operator on a Hilbert space $\mathcal{H}$, satisfying $A \geq 0$ (recall that this means $\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in \mathcal{H}$).

a. Show that $\sigma(A) \subset [0, \infty)$.

b. Let $x \in \mathbb{R}$ with $x \geq 0$. Show that there exists a $c \in [0, \infty)$ such that

$$\sqrt{x} = c \int_0^\infty \left(1 - \lambda^2 \left(x + \lambda^2\right)^{-1}\right) \mathrm{d}\lambda \; .$$

Hint: One can compute this explicitly or look for a shortcut.

c. Show that

$$\sqrt{A} = c \int_0^\infty \left(1 - \lambda^2 \left(A + \lambda^2\right)^{-1}\right) \mathrm{d}\lambda \; .$$

d. Let us now simplify to the finite-dimensional case $\mathcal{H} = \mathbb{C}^n$. Let $D > 0$ be a diagonal matrix (with respect to the canonical basis), let $v = (1, 1, 1, \ldots, 1)^T \in \mathbb{C}^n$ (also in the canonical basis) and let P_v be the rank-one orthogonal projection on v.

Let $A := D + P_v$**.**

Compute $\sqrt{A}$ as explicitly as possible.

Hint: Sherman-Morrison formula.