

# Metodi Matematici della Meccanica Quantistica

## Last Assignment

To be handed in on **Wednesday, January 8, 2025, before 23:59** via email  
(scanned or L<sup>A</sup>T<sub>E</sub>X) to [ngoc.nguyen@unimi.it](mailto:ngoc.nguyen@unimi.it).

### Problem 1: Wave Operators for Pseudo-Relativistic Particle (15 points)

Let  $\omega(p) := \sqrt{p^2 + 1}$  and consider the pseudo-relativistic Hamiltonian

$$H = H_0 + V, \quad H_0 = \sqrt{-\Delta + 1} := \mathcal{F}^{-1}T_\omega\mathcal{F}$$

where the potential  $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$  satisfies, for some  $\mu > 1$ , the decay  $|V(x)| \leq \text{const} \cdot |x|^{-\mu}$  for  $|x| > R$ . You can take for granted that  $H = H^*$  on  $D(H_0)$ .

Show that the wave operators  $\Omega_\pm$  exist.

*Hint:* Let  $D = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid \hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\}$ . Let  $\varphi \in D$  with  $\text{supp}(\hat{\varphi}) \subset \{|p| \geq \varepsilon\}$ . Then for all  $p \in \text{supp}(\hat{\varphi})$  we have by monotonicity

$$|\nabla\omega(p)| = \frac{|p|}{\sqrt{p^2 + 1}} \geq \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} =: 2\delta.$$

Decompose the potential into parts  $|x| \leq \delta t$  and  $|x| > \delta t$ .

### Problem 2: Abelian Limits (5 + 5 + 5 points)

a. Let  $H_0 = -\Delta/2$  in  $L^2(\mathbb{R}^3)$  and  $E \in \mathbb{R}$ . Show that

$$\text{s-lim}_{\varepsilon \downarrow 0} \varepsilon(H_0 - E + i\varepsilon)^{-1} = 0.$$

*Hint:* Use the Fourier transform and choose a convenient dense subspace.

b. Let  $\varphi : [0, \infty) \rightarrow X$  be continuous,  $X$  a Banach space and assume that  $\varphi_\infty := \lim_{t \rightarrow \infty} \varphi(t)$  exists. Prove that

$$\varphi_\infty = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} \varphi(t) dt.$$

c. Let  $H$  be a self-adjoint operator in  $L^2(\mathbb{R}^3)$  with  $D(H) = D(H_0)$ ,  $H_0 = -\Delta/2$ , and assume that the wave operator  $\Omega_+ = \text{s-lim}_{t \rightarrow \infty} e^{iHt} e^{-iH_0 t}$  exists. Assume furthermore

that asymptotic completeness holds, i. e.,  $\text{ran } \Omega_+ = \mathcal{H}_B^\perp$ , where  $\mathcal{H}_B$  is the closure of the span of the eigenstates of  $H$ .

Prove that for all  $\varphi \in \mathcal{H}$  we have

$$\Omega_+^* \varphi = \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty e^{-\varepsilon t} e^{iH_0 t} e^{-iHt} \varphi \, dt .$$

**Problem 3: Concatenation and Functional Calculus (15 points)**

Let  $A$  be a self-adjoint operator on a Hilbert space. Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be two measurable functions. Is it true that  $f(g(A)) = (f \circ g)(A)$ ? Provide a proof or a counterexample.

**Problem 4: An Integral Resolvent Representation (5+5+5+5 points)**

Let  $A = A^*$  be an operator on a Hilbert space  $\mathcal{H}$ , satisfying  $A \geq 0$  (recall that this means  $\langle \varphi, A\varphi \rangle \geq 0$  for all  $\varphi \in \mathcal{H}$ ).

- a. Show that  $\sigma(A) \subset [0, \infty)$ .
- b. Let  $x \in \mathbb{R}$  with  $x \geq 0$ . Show that there exists a  $c \in [0, \infty)$  such that

$$\sqrt{x} = c \int_0^\infty \left(1 - \lambda^2 (x + \lambda^2)^{-1}\right) \, d\lambda .$$

*Hint:* One can compute this explicitly or look for a shortcut.

- c. Show that

$$\sqrt{A} = c \int_0^\infty \left(1 - \lambda^2 (A + \lambda^2)^{-1}\right) \, d\lambda .$$

- d. Let us now simplify to the finite-dimensional case  $\mathcal{H} = \mathbb{C}^n$ . Let  $D > 0$  be a diagonal matrix (with respect to the canonical basis), let  $v = (1, 1, 1, \dots, 1)^T \in \mathbb{C}^n$  (also in the canonical basis) and let  $P_v$  be the rank-one orthogonal projection on  $v$ . Compute  $\sqrt{A}$  as explicitly as possible.

*Hint:* Sherman-Morrison formula.