

Metodi Matematici della Meccanica Quantistica

Assignment 1 (25/09/2026)

Problem 1: Banach Spaces

- a. Let X be a Banach space and $\tilde{X} \subset X$ a closed subspace. Show (in detail) that $\tilde{X}$ is a Banach space.
- b. Let X be a normed vector space and Y a Banach space. Show that $\mathcal{L}(X, Y)$ with the operator norm is a Banach space.

Problem 2: Boundary Conditions

Consider the Banach space $X = C([0, 1])$ with norm $\|f\| := \sup_{x \in [0, 1]} |f(x)|$.

Define operators $A_k : D_k \subset X \rightarrow X$ (for the two options $k \in \{3, 4\}$) by

$$(A_k f)(x) := f'(x)$$

with domains

$$\begin{aligned} D_3 &:= \{f \in C^1([0, 1]) \mid f(0) = f(1)\} && \text{(periodic boundary conditions)} , \\ D_4 &:= \{f \in C^1([0, 1]) \mid f(0) = 0 = f(1)\} && \text{(Dirichlet boundary conditions)} . \end{aligned}$$

Show that

- a. $\sigma(A_3) = \sigma_p(A_3) = 2\pi i\mathbb{Z}$,
- b. $\sigma(A_4) = \mathbb{C}$ and $\sigma_p(A_4) = \emptyset$.

Problem 3: Liouville's theorem

Prove the following theorem (you can use Liouville's theorem for complex-valued functions):

Let X be a Banach space. Let $L : \mathbb{C} \rightarrow \mathcal{L}(X)$ be an operator-valued entire function. If the operator norm of L is bounded, i.e., if there exists $M \in \mathbb{R}$ such that $\|L(z)\| \leq M$ for all $z \in \mathbb{C}$, then L is constant.

Problem 4: Polarization identity

For $\mathcal{H}$ a Hilbert space, prove the polarization identity:

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2) \quad \text{for all } x, y \in \mathcal{H}.$$