

Metodi Matematici della Meccanica Quantistica

Assignment 2 (09/10/2026)

Problem 1: Adjoint Operator

- a. Let $\mathcal{H}$ be a Hilbert space and A, B densely defined operators in $\mathcal{H}$.
Show that: If $A \subset B$, then $B^* \subset A^*$.
- b. Let $\mathcal{H}$ be a Hilbert space and A, B densely defined operators in $\mathcal{H}$.
Show that: If A is symmetric and B is a self-adjoint extension of A , then $A \subset B \subset A^*$.

Problem 2: Orthogonal Complement

Let $\mathcal{H}$ be a Hilbert space and $M \subset \mathcal{H}$ a subset. Prove the following facts:

- a. The orthogonal complement $M^\perp$ is a closed subspace.
- b. $M \subset (M^\perp)^\perp$. If M is a subspace: $(M^\perp)^\perp = \overline{M}$.

You may use that given a closed subspace $X \subset \mathcal{H}$, for every $x \in \mathcal{H}$ there exists a unique decomposition $x = x_1 + x_2$ with $x_1 \in X$ and $x_2 \in X^\perp$.

Problem 3: Operator with trivial adjoint

A countable family $(e_n)_{n \in \mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called **orthonormal basis** or **Schauder basis** if $\langle e_n, e_k \rangle = \delta_{n,k}$ and for all $x \in \mathcal{H}$ we have *as a series convergent with respect to the Hilbert space norm*

$$x = \sum_{n=0}^{\infty} e_n \langle e_n, x \rangle. \quad (1)$$

Remark: A basis that can represent any vector as *finite* linear combinations is called Hamel basis but is not commonly used in Hilbert spaces. One reason is that, if X is an infinite-dimensional Banach space, then any Hamel basis of X is uncountable.

Example: For $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$, in $\mathcal{H} = L^2((0, 2\pi))$, (1) is the Fourier series.

Now let $\mathcal{H} := L^2(\mathbb{R})$ and $(e_n)_{n \in \mathbb{N}}$ an arbitrary orthonormal basis of $\mathcal{H}$. Define an operator $A : \mathcal{D} \subset \mathcal{H} \rightarrow \mathcal{H}$ by $\mathcal{D} := C_0^\infty(\mathbb{R})$ and

$$Af := \sum_{n=0}^{\infty} f(n) e_n. \quad (2)$$

- a.** Show that: For any $g \in \mathcal{H}$, $g \neq 0$, the mapping $f \mapsto \langle g, Af \rangle$ is not continuous as a function from $(\mathcal{D}, \|\cdot\|_{\mathcal{H}})$ to $\mathbb{C}$.
- b.** Show that $D(A^*) = \{0\}$.