Metodi Matematici della Meccanica Quantistica

Assignment 5

Problem 1: Method of Stationary Phase

Let $\omega \in C^\infty(\mathbb{R}^n)$ and $u \in C_0^\infty(\mathbb{R}^n)$. Let G be an open neighbourhood of $\{\nabla\omega(k) \mid k \in \mathrm{supp}(u)\}$. For simplicity consider only $n = 1$ in the following (but note that the result holds for any $n \in \mathbb{N}$).

Show that for all $m \in \mathbb{N}$ there exists $c_m \in \mathbb{R}$ such that

$$\left| \int_{\mathbb{R}^n} e^{i(k\cdot x - \omega(k)t)} u(k) \mathrm{d}k \right| \leq \frac{c_m}{(1+|t|)^m} \quad \text{for all } x \text{ and } t \text{ satisfying } \frac{x}{t} \notin G.$$

Heuristics: This quantifies the intuition that positive and negative parts of the integral cancel if the phase is rapidly oscillating. The integral "survives" only when the phase is stationary (when $x/t \in G$).

Hint: Let $\phi(k) := k\frac{x}{t} - \omega(k)$. Use $e^{i\phi(k)t} = \left(\frac{1}{i\phi'(k)t}\frac{\mathrm{d}}{\mathrm{d}k}\right)^m e^{i\phi(k)t}$ and integration by parts.

Problem 2: Wave Operators for Pseudo-Relativistic Particle

Let $\omega(p) := \sqrt{p^2+1}$ and consider the pseudo-relativistic Hamiltonian

$$H = H_0 + V\ , \quad H_0 = \sqrt{-\Delta + 1} := \mathcal{F}^{-1} T_\omega \mathcal{F}$$

where the potential $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies, for some $\mu > 1$, the decay $|V(x)| \leq \mathrm{const} \cdot |x|^{-\mu}$ for $|x| > R$. You can take for granted that $H = H^*$ on $D(H_0)$.

Show that the wave operators $\Omega_\pm$ exist.

Hint: Let $D = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid \hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\}$. Let $\varphi \in D$ with $\mathrm{supp}(\hat{\varphi}) \subset \{|p| \geq \varepsilon\}$. Then for all $p \in \mathrm{supp}(\hat{\varphi})$ we have by monotonicity

$$|\nabla\omega(p)| = \frac{|p|}{\sqrt{p^2+1}} \geq \frac{\varepsilon}{\sqrt{\varepsilon^2+1}} =: 2\delta\ .$$

Decompose the potential into parts $|x| \leq \delta t$ and $|x| > \delta t$.