Metodi Matematici della Meccanica Quantistica

Assignment 6

Problem 1: Square Well

Let $a, V_0 > 0$, and consider

$$H := -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - V_0\chi_{(-a,a)} \qquad \text{on } D(H) = H^2(\mathbb{R}).$$

Suppose that $0 \neq \varphi \in H^2(\mathbb{R})$ satisfies $H\varphi = E\varphi$ with $E < 0$, and define $\kappa := \sqrt{-2E}$.

a. Show that φ has a C^1–representative and is smooth on each of the intervals

$$(-\infty, -a), \qquad (-a, a), \qquad (a, \infty).$$

Show that on the two exterior intervals, for some constants $A_-, A_+ \in \mathbb{C}$, we have

$$\varphi(x) = \begin{cases} A_- e^{\kappa x}, & x < -a, \\ A_+ e^{-\kappa x}, & x > a. \end{cases}$$

b. Show that both A_- and A_+ are nonzero. Deduce that, for $\beta \geq 0$,

$$e^{\beta|x|}\varphi \in L^2(\mathbb{R}) \quad \Longleftrightarrow \quad \beta < \kappa.$$

c. Show that the right/left–sided second derivatives satisfy

$$\varphi''(a+) - \varphi''(a-) = 2V_0\varphi(a),$$

and conclude that $\varphi \notin C^2(\mathbb{R})$. Compare this result to Theorem 7.5.

Problem 2: Compactness vs. Relative Compactness

Let $H_0 := -\Delta/2$ on $H^2(\mathbb{R}^n)$. Consider the unit cube $Q = (0,1)^n$ and let M be the multiplication operator by the characteristic function χ_Q on $L^2(\mathbb{R}^n)$:

$$(Mu)(x) := \chi_Q(x)u(x).$$

a. Consider

$$u_k(x) := \chi_Q(x)e^{2\pi i k x_1}, \qquad k \in \mathbb{N}.$$

Show that $(u_k)_k$ is a sequence whose elements are all orthonormal to each other. Show that $(Mu_k)_k$ has no norm-convergent subsequence. Conclude that M is not compact.

b. Show that M is relative compact with respect to H_0.